Abstract Algebra: Teachers' Guide

Slightly edited from *Algebra: Themes, Tools, Concepts,* by Anita Wah and Henri Picciotto
More info: http://www.MathEducation.page/abs-alg

These lessons appeal to many students' imagination and show that mathematics can be interesting when it is frivolous, as much or more as when it is serious or useful.

The properties of zero and one in addition and multiplication, respectively, are not very interesting to students (or teachers) in the context of the real numbers. After eight or more years of exposure to the fact that 2 + 0 = 2, it loses its appeal. In these lessons, the identity elements for mathematical structures other than the real numbers provide a fresh environment to think about one and zero. Likewise, opposites and reciprocal are approached in an unfamiliar context.

The mathematical content underlying these lessons is not a traditional part of the pre-college curriculum. Yet, the ideas of identity and inverse elements and the mathematical structures known as groups and fields are the backbone of arithmetic and algebra. The approach taken here proves that these ideas do not have to be presented in an overly abstract and dull manner, as was too often the case with the New Math of the sixties and seventies.

1. Math on Another Planet: Small Pockets

It would be a good idea for you to work out the problems before attempting them with the class. (They are not difficult, just unusual.)

This lesson requires a lot of reading. Perhaps students should be encouraged to take turns reading aloud in their groups.

After introducing the basic rules, you may have students practice them with the help of the **ducats.pdf** document and counters of some sort (such as pennies or centimeter cubes.) The idea is that two counters in the **d** area can be traded for one counter in the **e** area, etc. (If you're familiar with "exploding dots", this is a sort of binary version of that, but it is also circular.) Starting this way makes the whole thing much more accessible.

#1 requires a certain amount of careful computation. The computations are almost purely algebraic, and involve very little work with numbers. This exploration previews all of #2-5.

2. Math on Another Planet: A Long Month

There are computational shortcuts possible. For example, in #6,

Mo+Mo → 5+5 → 10 → Sa.
So, Mo+Mo+Mo = Mo+Sa → 5+3=8 → Th.

In #9, point out that calendar zero, just like zero among the real numbers, does not have a reciprocal. Also, zero times any number equals zero.

Both #6-7 and #11 point out the cyclical nature of this mathematical structure. A subtle point, probably too difficult for students to discover or even understand, is that the addition table for Calendar Math has *exactly* the same underlying structure as the addition table of ducats, ecus, and florins, in the previous lesson.

For #12, you may ask students to investigate whether the commutative, associative, and distributive laws hold in Calendar Math. You can also introduce the term *identity element* to describe 0 for addition, and 1 for multiplication, and ask that these be included in the summary, along with information on opposites and reciprocals.

3. Letter Strings

This activity, while not particularly difficult, will feel unfamiliar. It works best as a group activity in class.

The YZ Game

You can introduce the game by writing a very long string of Ys and Zs on the board and having students suggest ways to simplify it according to the rules. With the next string, you can ask them to predict the outcome of simplifying.

#2: the elements are E, Y, YY, Z, YZ, YYZ

#6: it is interesting to note that the powers of YZ and YYZ span all six elements in the group.

The yz Game

This is quite a bit more challenging, because of the absence of the commutative law. It would be best to have the students prepare group reports for #12.

Note that yyz = yyyzy = zy, and zyy = yzyyy = yz

The elements of the group are e, y, yy, z, yz, and zy. yz and zy are not equal! A consequence of this is that it is quite difficult to correctly predict what a long string will simplify to.

Another difference with the commutative case is that no element's powers span the whole group.

4. Smooth Moves

It is helpful for the triangles to be made of some kind of stiff cardboard. Paper will not work nearly as well. Having a class set of ready-made triangles will save some fussing. If you plan to assign homework from these sections, students will have to make their own triangles.

Patterns for the triangles can be found in the file **triangle.pdf**.

To make sure everyone understands how to do moves in succession, you may want to have all students do the example, and perhaps other problems like it, at the same time, and make sure everyone gets to the same endpoint. The most common mistake is in executing f_2 and f_3, which students often confuse with f_1.

Once students are familiar with the basics of this activity, a fun follow-up is to get three student volunteers, call them A, B, and C. (If you have nametags for them to wear, use them.) Have them hold hands, making a triangle, and place markers labeled 1, 2, and 3 on the floor. Then have students call out various moves (a, f_1, etc.) and have A, B, and C carry out the moves, without ever letting go of each others' hands. The turns are easy to carry out, but the flips require some interesting maneuvers.

5. The Algebra of Moves

In #1, you may want to demonstrate how to fill one or two cells of the table, taking care to emphasize which move is done first, and which is done next. To yield a correct table, this must be done consistently throughout.

6. Group Theory

This is the last lesson of this packet on group theory and abstract algebra. It is about groups of numbers that have the same structure as the quite varied groups seen in the previous lessons. The idea that apparently different things can share a common structure may be the most fundamental concept in mathematics, and certainly in algebra.

Because it is so abstract and number-based, most of the lesson should probably be done in class. The first part of it is essentially a lesson in number patterns and arithmetic.

Mod Clocks

The mod clock provides an example of a function from number theory. It behaves quite differently from most of the functions students see in algebra, and therefore it helps broaden their horizons. Another benefit of this section is the review of divisibility patterns.

A good way to introduce this lesson is to play "What's my rule?" (See teachers' notes for Chapter 2, Lesson 7.) This can be played for #1 and #3. Mod 5, 10, and 2 outputs can easily be predicted from the last digit of the input. Mod 9 and 3 inputs depend on the sum of the digits, though you should let students discover this for themselves if they can. (The mod 9 pattern is easier to see than the mod 3 pattern.)

The connection with the remainder of a division by the mod number can also be discussed. The "addition" and "multiplication" tables will be reminiscent of the tables made in the previous group theory lessons. Students will need help in getting started. (See the blank tables in the back of this Teachers' Book.) A comparison of the tables may lead to the discovery of interesting patterns:

- if the operation is commutative, the table is symmetric around its main diagonal
- the identity element's row and column look like the elements outside the table
- in some cases, no elements are repeated in any row or column

Groups

This is where we give the formal definition of a group. Merely combining a set with an operation does not guarantee a group. For example, the natural numbers do not constitute a group with any operation, because of the lack of inverse elements. (For addition, you need negative numbers, and for multiplication you need fractions.)

Because the definition is abstract, it is best to discuss it in the context of an example, such as the one given in #4. A discussion of #4-7 can be supplemented with a look back at the groups studied in previous chapters in the light of the new terminology.

#8: For examples students can look back at the lessons listed above, especially for the non-commutative case, of which there are no cases in this lesson (instead, see "Letter Strings" and "Smooth Moves".) You can help them get started by discussing an example in class, and discussing where others can be found.

If you want to do more on abstract algebra, one possible investigation is to explore which of the mod clocks yield groups. It turns out that any number yields a group for $\oplus$, but only prime numbers yield groups for $\otimes$ (and this only once zero has been removed from the set.)

At a more advanced level (11th or 12th grade), you may use Richard Brown's book on *Transformational Geometry*, (1973, Silver, Burdett, and Ginn, Inc. reissued by Dale Seymour Publications, and now seemingly out of print) to explore groups with your precalculus students.

See also the follow-up lessons about fields and isomorphism on this site.