CHAPTER 4

The spiral curve of a West African chameleon's tail

Coming in this chapter:

Exploration

- Find as many functions as possible whose graphs go through the origin.
- Find as many functions as possible whose output is 5 when the input is 2.

INTERPRETING GRAPHS

LESSON 4.1

A 100-Mile Trip

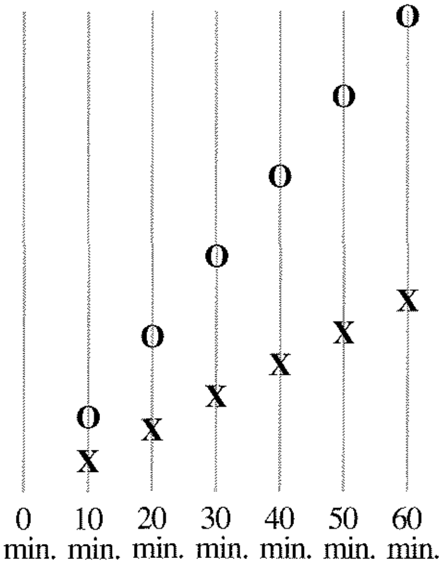
You will need:

graph paper

1. By which of these methods do you think a person could travel 100 miles in one day? Explain how you arrive at your guess.

walking	running
bicycling	ice skating
riding a scooter	riding in a car
riding in a helicopter	

2. Ophelia and Xavier are traveling along a road. If you could view the road from above and make a sketch of what you saw every ten minutes, your sketches might look something like the figure below.
   a. Which person (O or X) is traveling faster?
   b. If the entire length of the road is six miles, can you figure out approximately how fast each person is traveling? Explain.

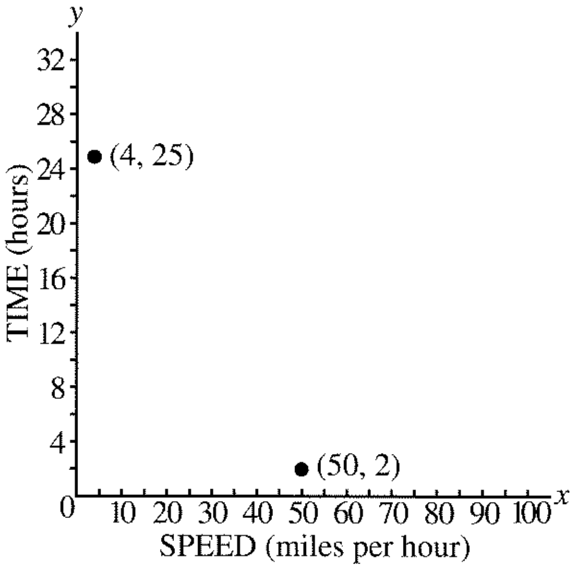


TIME VS. SPEED

3. Copy and complete this table showing how many hours it would take each person to travel 100 miles.

Person	Mode of Travel	Speed (mph)	Time (hours)
Abe	walking	4	25
Al	van	50	
Bea	skating	10	
Gabe	scooter	30	
Lara	helicopter	100	
Lea	bike	25	
Reg	running	8	

4. Copy and complete the graph that shows how long it would take for each person to make the 100-mile trip.

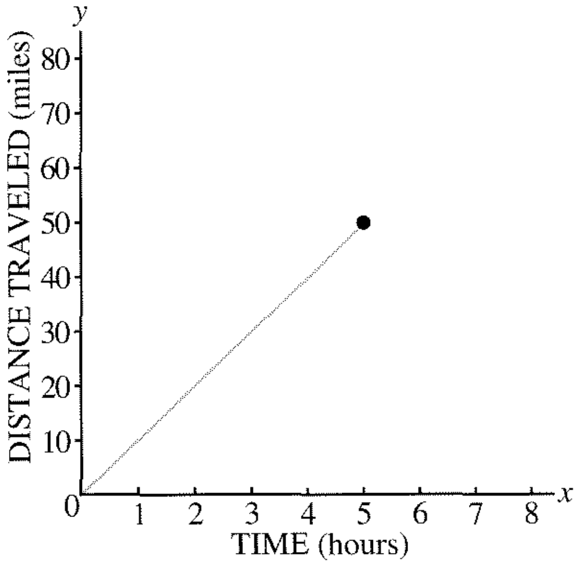


5. **Generalization**

a. What pattern do you notice in the table?

b. How long would it take for someone who travels at a constant speed of S miles per hour to cover 100 miles?

DISTANCE VS. TIME



6. The graph shows Bea's progress on the trip. It shows that after 5 hours of roller-skating she had traveled 50 miles.

a. Copy the graph onto graph paper. Use a whole piece of graph paper. You will be adding more to this graph.

b. One of the points on the graph is (5, 50). Mark and label three more points on the graph of Bea's progress.

7. In this lesson we are assuming everyone travels at a constant speed. How valid is this assumption? For each mode of travel what might make it impossible to travel at a constant speed? Explain.

The table shows how long it took for Abe to go certain distances.

Abe's Progress

Time (hours)	Distance (miles)
1	4
2	8

8. a. Copy and complete the table up to 20 miles.

b. For this problem, use the same axes you used for Bea. Plot and label the points from the table in part (a).

c. Connect the points with a straight line. Then find and label a point that is on the line but not in your table. Interpret the coordinates of the point in terms of this problem.

9. Make a table like the one you made for Abe showing Gabe's progress on his scooter and Al's progress in the van. Make graphs of their progress on the same axes you used to show Abe's and Bea's progress. Label the four different lines.

10. Use your graphs to help you answer these questions. If Bea and Abe start out at the same time,

a. how far apart will they be after one hour?

b. how far apart will they be after two hours?

11. **Generalization** Look for a pattern. How far apart will Abe and Bea be after H hours? Explain.

12. Mrs. Gral was traveling at a constant speed. She started at the same time as Abe, and was two miles ahead of him after one hour.

a. Add a graph of Mrs. Gral's progress to your axes.

b. How far ahead was Mrs. Gral after two hours?

c. After three hours, how far was Mrs. Gral behind Bea?

d. How fast was Mrs. Gral going? What mode of travel do you think she was using?

13. Summary

a. How does the mode of travel affect the steepness of the line? Explain.

b. What is the meaning of points on two of the graphs that have the same x-coordinate but different y-coordinates?

c. What is the meaning of the vertical distance between two lines for a given value of x?

DISTANCE VS. SPEED

14. Using the same speed data, figure out how far each person could travel in two-and-a-half hours. Make a table and a graph showing speed on the horizontal axis and distance on the vertical axis.

15. 🔑 How would the graph be changed if the travel time was greater? Less? Explain.

SPEED BY GRAPHS AND FORMULAS

16. Summary Each graph in this lesson gives information on how fast people travel, but it does it in a different way. Explain.

17. Generalization If someone is traveling at a constant speed of S miles per hour, for a distance of D miles, and takes T hours, what is the relationship between S, D, and T? Write this relationship in more than one way.

DISCOVERY *FRAMING PHOTOGRAPHS*

A photograph is mounted on a background which sticks out one inch on each side. The width of the photo is two inches and the height is three inches.

18. a. Sketch the photo and its frame.

b. What are the dimensions of the frame?

c. Are the photo and frame similar rectangles? Explain.

19. The photo needs to be enlarged so it will fit in a frame having a height of 12 inches. Again, the width of the frame is to be one inch. Find the dimensions of the enlarged photo and its frame. Of course the photo cannot be distorted!

20. Is the frame for the enlarged picture similar to the picture? Is it similar to the original frame? Explain.

LESSON

4.2

Points, Graphs, and Equations

You will need:

graph paper

PATTERNS FROM POINTS

1. a. Draw a pair of axes and plot these points.

x	*y*
0	-1
-2	-3
-5	-6
5	4

b. Study the table and your graph. Describe the relationship between the *x*-value and *y*-value of each pair.

c. Use the pattern you found to add more points to your table and graph.

d. Write an equation that tells how to get the *y*-value from the *x*-value.

2. Repeat problem 1 for each of these tables.

a.

x	*y*
4	-8
1	-2
-3	6
0	0

b.

x	*y*
-3	-3
5	-3
-6	-3
-1	-3

c.

x	*y*
6	4
12	-2
-1	11
3	7

GRAPHS FROM PATTERNS

3. For each description below, make a table of at least five (*x*, *y*) pairs that fit it. Then graph the (*x*, *y*) pairs. Use a separate coordinate system for each graph.

a. The *y*-coordinate is always equal to the *x*-coordinate.

b. The *y*-coordinate is always four less than the *x*-coordinate.

c. The *y*-coordinate is always one-half of the *x*-coordinate.

d. The *y*-coordinate is always the opposite of the *x*-coordinate.

e. The *y*-coordinate is always the square of the *x*-coordinate.

EQUATIONS FROM PATTERNS

4. For each description in problem 3, find an equation that describes the relationship between *x* and *y*. Write the equations on your graphs.

5. a. Make a table of four number pairs (*x*, *y*) that have this property: The sum of *x* and *y* is always 6.

b. Graph these (*x*, *y*) pairs.

c. Connect the points with a straight line.

d. Write the relationship between *x* and *y* as an equation.

6. a. Using fractions and negative numbers, write two more (*x*, *y*) pairs having the property that the sum of *x* and *y* is 6. Do these points lie on the line?

b. Choose a point that is not on the line. Do its (*x*, *y*) coordinates add up to 6?

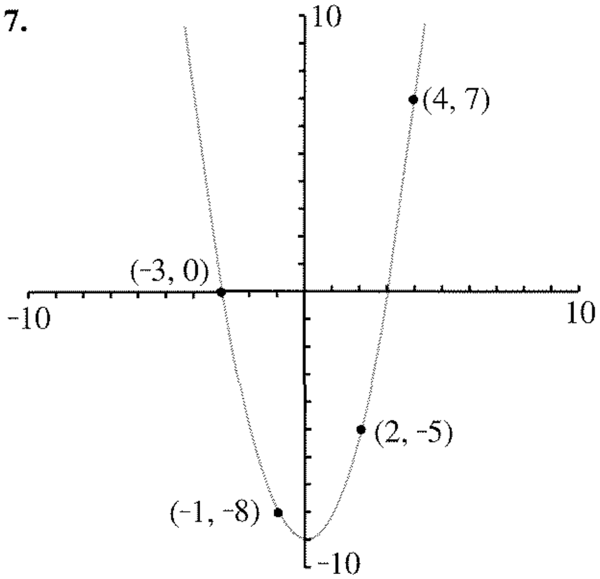
c. Write any number pair (*x*, *y*) whose sum is not 6. Find this point. Is it on the line you drew?

EQUATIONS FROM GRAPHS

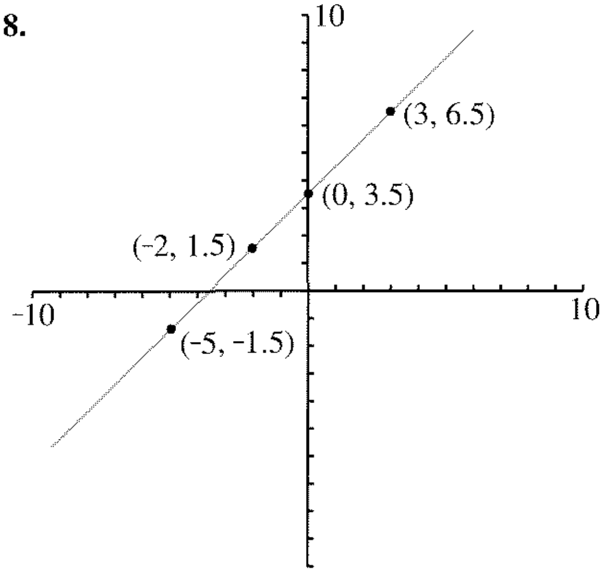
On each graph below, four points are labeled. For each graph:

a. Make a table of the (x, y) pairs and look for a relationship between x and y.
b. Add three more points to the table, making sure each one does belong on the graph.
c. Write an equation describing the relationship between x and y.

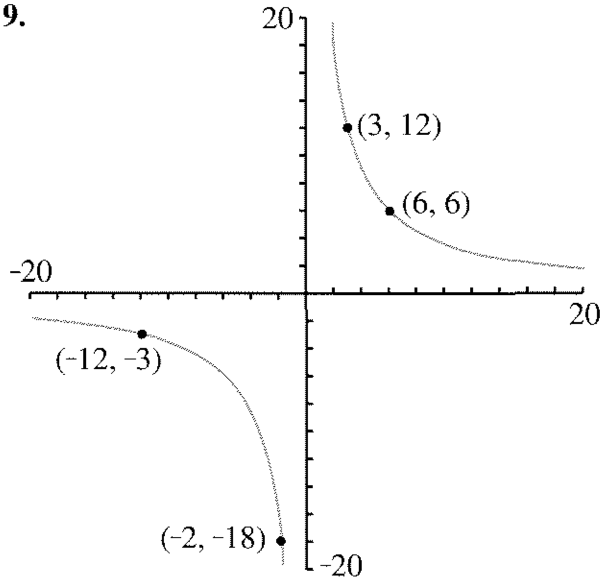
7.



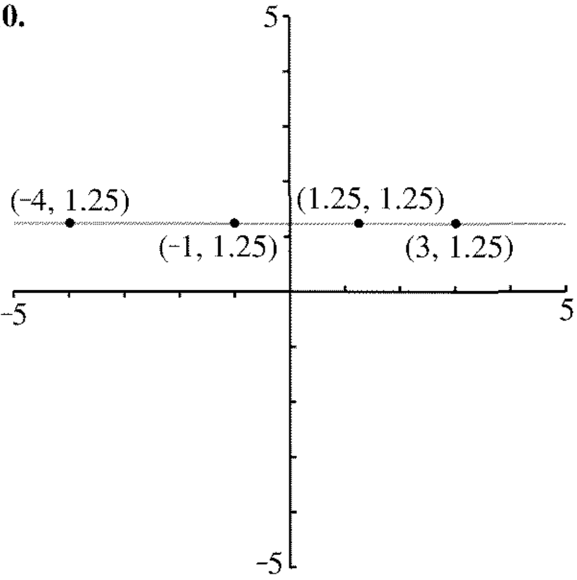
8.



9.



10.



POINTS AND EQUATIONS

The following questions are about the graph of the function $y = 4x + 5$. Try to answer the questions without graphing.

11. Is the point (7, 32) on it? Explain.

12. The point $(3, y)$ is on it. What is y? Explain.

13. The point $(x, 6)$ is on it. What is x? Explain.

LESSON 4.3

Polynomial Functions

You will need:

graph paper

Definition: A *polynomial function* is a function of the form $y =$ a polynomial.

1. **Exploration** Which of these polynomial functions do you think have graphs that are straight lines? Which have curved graphs? Explain why you think so.
   a. $y = x^2$ b. $y = 2x - 1$
   c. $y = 2x^2$ d. $y = x^3$

ORDER OF OPERATIONS

2. Make a table of at least eight (x, y) pairs for each function. Use negative numbers and fractions as well as positive whole numbers in your tables. Then make a graph from each table. Label each graph with its equation. You will need to refer to these graphs later.
   a. $y = x^2$ b. $y = x^3$

To make a table of values for graphing $y = -x^2$, we have to know what the expression $-x^2$ means. Does it mean *square x, then take its opposite* or *take the opposite of x, then square it*? Which operation should be done first?

To avoid this kind of confusion, mathematicians have agreed on the following rule.

Rule: *Exponentiation should be performed before other operations.* To change this order, we have to use parentheses.

Examples:
- $-x^2$ means *square x, then take the opposite.*
- $(-x)^2$ means *take the opposite of x, then square the result.*

3. Make a table of at least eight (x, y) pairs for each function. Use negative numbers and fractions as well as positive whole numbers in your tables. Then make a graph from each table. Label each graph with its equation.
   a. $y = (-x)^2$ b. $y = -x^2$

4. Compare your graphs in problem 3 with the graph of $y = x^2$. Explain what you observe.

5. Graph these polynomial functions.
   a. $y = -x^3$ b. $y = (-x)^3$

6. Compare your graphs in problem 5 with the graph of $y = x^3$. Explain what you observe.

DEGREE

Definition: The degree of a polynomial function in one variable is the highest power of the variable that appears in the polynomial.

Examples: $y = x^3$ and $y = x^2 + 2x^3$ are both third-degree polynomial functions. The equation $y = 2x$ is first-degree, and the equation $y = 1$ is zero-degree.

7. What is the degree of each of these polynomial functions?
   a. $y = 5 + x^2 - x$
   b. $y = 4x^3 - 3x^2 + 5$
   c. $y = 45$

8. Make a table of at least eight values for each third-degree function. Use negative numbers and fractions as well as positive whole numbers in your tables. Then make a graph from each table.
   a. $y = 2x^3$ b. $y = x^3 + 1$
   c. $y = -x^3 - 2$

9. Repeat problem 8 for these second-degree functions.
   a. $y = x^2 - 1$ b. $y = -3x^2$
   c. $y = -x^2 + 2$

10. Graph these first-degree functions.
   a. $y = 5x$ b. $y = x$
   c. $y = -2x + 1$

11. Graph these zero-degree functions.
   a. $y = 4$ b. $y = -3$
   c. $y = 0$

THE EFFECT OF DEGREE

12. Tell whether each sentence (a-b) could describe the graph of a zero-degree, first-degree, second-degree, or third-degree polynomial function. More than one answer may be possible for each description.
   a. The graph is a straight line.
   b. The graph is a curve.

13. Repeat problem 12 for these descriptions.
   a. The graph goes through the origin.
   b. The graph never crosses the x-axis.
   c. The graph never crosses the y-axis.

14. Repeat problem 12 for these descriptions.
   a. The graph passes through quadrants I and III only.
   b. The graph passes through quadrants II and IV only.
   c. The graph passes through quadrants I and II only.

15. **Summary** How does the degree of the equation affect its graph? Write a summary explaining everything you know about this.

16.
   a. Make a table of values and graph the function $y = 24/x$.
   b. Is this a polynomial function? Explain.

REVIEW *POSSIBLE OR IMPOSSIBLE?*

17. Decide whether each of the following situations is possible or impossible. If it is possible, give an example. If it is impossible, explain *why* it is impossible. Can you subtract
   a. a negative number from a negative number to get a positive number?
   b. a negative number from a negative number to get a negative number?
   c. a negative number from a positive number to get a positive number?
   d. a negative number from a positive number to get a negative number?
   e. a positive number from a negative number to get a negative number?
   f. a positive number from a negative number to get a positive number?

LESSON 4.4

Graphs Through Points

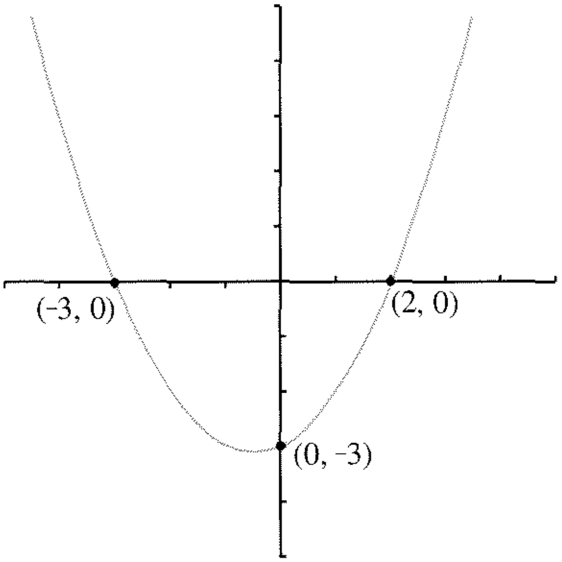
You will need:

graph paper

graphing calculator (optional)

INTERCEPTS

Definitions: The *y-intercept* of a graph is the point where the graph crosses the y-axis. The *x-intercept* of a graph is the point where the graph crosses the x-axis.



Example: The curve in the figure above has y-intercept (0, -3), and x-intercepts (-3, 0) and (2, 0).

For problems 1-5:

- a. Guess the coordinates of the x- and y-intercepts (if you think they exist).
- b. *On graph paper* draw the graph described.
- c. Check the correctness of your guess.

1. A line is parallel to the y-axis and passes through the point (2, -3).
2. A line passes through the origin and the point (2, -3).
3. The sum of every (x, y) pair on the line is 8.
4. The line passes through the points (2, -3) and (3, -2).
5. To get the y-coordinate, square the x-coordinate and add 1.

POINTS ON AND OFF GRAPHS

6. Bea thinks that $8 - 2x$ means *multiply x by 2 and subtract the result from 8.* Lea thinks it means *subtract 2 from 8 and multiply the result by x.* Who is right? Explain.
7. Which of these points do you think will lie on the graph of $y = 8 - 2x$? Explain.

a. (2, 4)	b. (2, -4)
c. (0.5, 6)	d. (0.5, -6)
e. (-1, -10)	f. (-1, 10)

For the remaining problems in this lesson (8-23), use a graphing calculator if you have one. Otherwise, use graph paper.

8. a. Graph $y = 8 - 2x$.
   - b. Use your graph to check your answers to problem (a).
   - c. Write both coordinates of the x-intercept of $y = 8 - 2x$.
   - d. Write both coordinates of the y-intercept of $y = 8 - 2x$.

Definition: If two graphs share a point, they are said to *intersect* at that point.

9. a. On the same coordinate system, graph $y = 2x - 8$.
   - b. Do your two graphs intersect at any point? If so, where?

Follow these instructions for problems 10 through 12 below.

a. Make tables of values for the two functions given. Then graph them on the same pair of axes. Label at least three points on each graph.
b. Find and label a point that is not on either graph.
c. Find and label a point that is on both graphs (if there is one).
d. Find and label a point that is in the region between the two graphs.
e. Find and label a point that is neither on nor between the graphs.

10. $y = 2x$ and $y = 0.5x$

11. $y = x$ and $y = x + 2$

12. $y = x^2$ and $y = x^2 - 3$

13. For problems 10-12, find an equation whose graph is entirely contained between the two given graphs.

FIND AN EQUATION

In problems 14-17, find the equation of any graph that satisfies the characteristics given.

14. A second-degree function whose graph passes through the point (0, 0)

15. A second-degree function whose graph passes through the point (0, 1)

16. A third-degree function whose graph passes through the point (0, -1)

17. A first-degree function whose graph passes through the point (-1, -1)

18. a. Write any equation whose graph contains the point (1, 2).
b. Write any other equation whose graph passes through the point (1, 2).
c. Graph the two equations. Where do they intersect?

19. **Report** Write a report explaining the answers to these questions. Use examples in your explanations.
a. Given an equation, how can you figure out which points lie on its graph?
b. Given a point and an equation, how can you tell whether or not the point lies on the graph of the equation?

GRAPHS THROUGH THE ORIGIN

20. Which of the following equations have graphs that go through the origin? How could one tell without actually graphing them?
a. $y = 2x - 6$ b. $y = x^2 - x$
c. $y = -x^3 - 4$

21. Give three equations (one each of first, second, and third degree) that satisfy each of these two given conditions.
a. The graph will pass through the origin.
b. The graph will not pass through the origin.

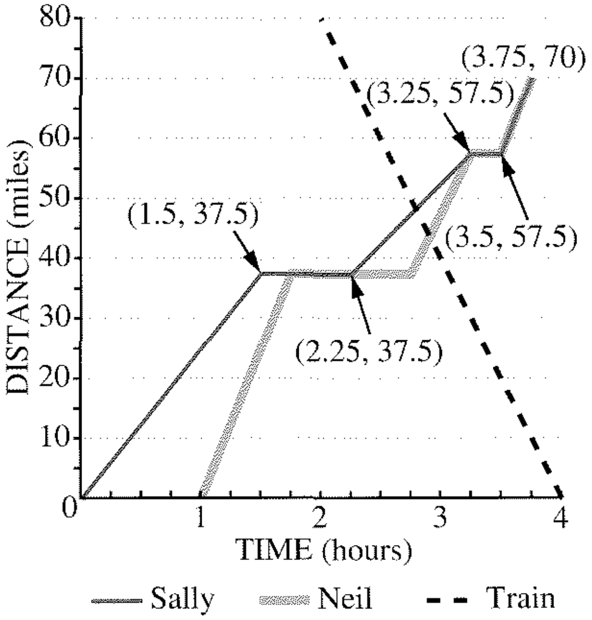
22. Write the equation of a graph that lies in quadrants I and III *only* and
a. passes through the origin;
b. does not pass through the origin.

23. **Summary** Explain how you can tell from an equation whether or not its graph goes through the origin. Give some examples.

THINKING
WRITING

4.A The Bicycle Trip

Sally is riding her bike on a trip with her bicycle club. She left the staging area in Chapley at 10 A.M. and took a break at a rest area located about halfway to the final destination of Berkhill, 70 miles away. Neil is driving the sweep vehicle, a van with food, water, first aid, and a bicycle rack. The distance-time graph below shows their progress. There are train tracks along the road. The progress of a train is also shown on the graph.



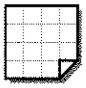
1. Compare Sally's and Neil's progress. Who left first? Where did she or he stop? What happened at the end? What was the total distance covered?
2. Including the origin, the coordinates of six points on Sally's graph are given. Describe her ride between consecutive points.
    a. At what time did each leg of her trip start and end? How far did she ride each time? How long did it take? How long were her breaks?
    b. How fast was she going during each leg of the trip?
3. a. If you were to guess about which part of the trip was downhill or uphill, what would you guess? Why?
    b. How else might one account for the different speeds?
4. How fast did Neil drive in each leg of his trip?
5. Describe the train's progress. Which way was it going? Where and when did it pass Sally and Neil?
6. Where were Sally, Neil, and the train at 12:30 P.M.?
7. At what times were Sally, Neil, and the train 20 miles from the staging area?
8. The equation of the train's motion is $D = 160 - 40t$.
    a. Choose three points on the train's graph and check that their coordinates satisfy the equation.
    b. Do any points in Sally's and Neil's graphs satisfy the train's equation? If so, which ones?
9. **Summary**
    a. In a distance-time graph, what does it mean if two points are on the same horizontal line? On the same vertical line?
    b. As you go from left to right on the graph, what is the meaning of a part that goes up? Down? What is the meaning of a horizontal segment? Why is a vertical segment impossible?
    c. What is the significance of a point that belongs to the motion graphs of two different people?
10. **Report** Tell the story of the bicycle trip. Use information you gathered from the graph. Make guesses about the trip. Include a graph for Irva, another member of the bicycle club. She too left at 10 A.M. and stopped at the rest area.

LESSON 4.5

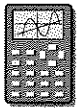
Lines Through the Origin

You will need:

graph paper



graphing calculator (optional)



Definition: Since the graphs of first-degree equations are straight lines, these equations are also called *linear* equations.

1. Predict whether or not the graph of each linear equation will pass through the origin. Explain how you know, using graphs or calculations.

a. $y = 4 - 2x$ b. $y = -2x$

c. $y = 2x$ d. $y = 2x - 4$

2. Write two linear equations which you think will have graphs through the origin. Explain your reasoning.

RATIO

Lara and Lea were arguing about points and graphs. Lea said, "If the point (1, 4) lies on the line, then the point (2, 8) must also lie on the line." Lara showed her that she was wrong by drawing three lines.

3. On graph paper, draw a line that goes

a. through both points;

b. through (1, 4) but not through (2, 8);

c. through (2, 8) but not through (1, 4).

4. Of the three lines you drew in problem 3, which goes through the origin?

5. a. Plot and label at least three more points that are on the line through (1, 4) and (2, 8).

b. Find the equation of the line through (1, 4) and (2, 8).

6. 🔑 Plot these eight points on the same axes. Label them with their coordinates.

(1, 2)	(-1, -2)	(1, -2)
(-1, 2)	(3, 6)	(-3, -6)
(6, 3)	(6, -3)	

a. Draw a line connecting each point with the origin. Which points lie on the same line through the origin?

b. Explain how to find the equations of the lines you drew.

Definition: The *ratio* of *a* to *b* is the result of the division *a*/*b*.

Example: The ratio of 6 to 3 is 6/3 or 2, while the ratio of 3 to 6 is 3/6, or 1/2, or 0.5.

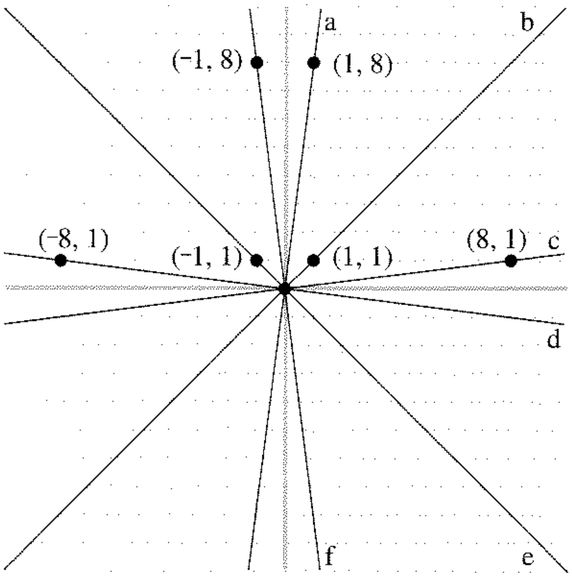
7. a. Write two (x, y) pairs for which the ratio of y to x is 1/3.

b. Plot these two points and graph the straight line through them. Find the equation of the line.

c. Write two (x, y) pairs for which the ratio of y to x is 3.

d. Plot these two points and graph the straight line through them. Find the equation of the line.

8. For each line in the graph below, find three points on the line. Then find an equation for the line.



9. Explain how you can find more points on the same line through the origin as (4, 5) without drawing a graph. Then check by graphing the line. Find the equation of the line.

Lea noticed that for the points (1, 4) and (2, 8) the ratio of the y-value to the x-value was the same. That is, $4/1 = 8/2$. She guessed that (100, 400) will lie on the same line through the origin because the ratio of the y-value to the x-value is also 4.

10. Tell whether or not you agree with Lea, and why.

11. Find a point whose coordinates have the same ratio of y to x as the point (4, 12). Does this point lie on the same line through the origin as (4, 12)? If so, find the equation of this line.

12. a. Graph the line through (-1, 2) and (3, 4).
   b. Is the ratio of 5 to -10 equal to the ratio of -1 to 2?
   c. Is the point (5, -10) on the line? Explain why or why not.

13. **Generalization**
   a. What would be the ratio of the coordinates of points on the line through the origin and the point (a, b)? Explain.
   b. If $b/a = d/c$, what can you say about the line joining (a, b) to (c, d)? Explain.

14. **Summary** Explain what ratio has to do with lines through the origin.

SPEED

The table shows the amount of time it took several people to travel the distances given.

Person	Time (hours)	Distance (kilometers)
A	3	80
B	7	140
C	12	320
D	1	30
E	2	30
F	1	20
G	5	150

15. a. Draw a pair of axes and label the vertical axis *distance* and the horizontal axis *time.* Plot and label the points in the table. Draw lines connecting each point with the origin.
   b. Which points lie on the same line through the origin?

16. Use the table and your graph to answer these questions.
   a. Which people are traveling at the same speed?
   b. Who is traveling faster, *A* or *B*?
   c. How far will *A* have traveled in four hours?

17. a. H has been traveling two hours at the same speed as G. Add H to your graph.

b. I have been traveling four hours at the same speed as A. Add me to your graph.

18. J is traveling faster than B but more slowly than D. Draw one possible distance-time graph showing J's progress.

19. Each line you drew has an equation that relates distance to time. Find these equations and add them to your graph.

20. **Summary**

a. Explain how one can think of speed as a ratio.

b. If you are given time and distance for two travelers, explain how to use calculations or graphs to compare their speeds.

DISCOVERY *HAPPY NUMBERS*

Take the number 23.
Square each digit and add.
$2^2 + 3^2 = 13$
Repeat this process.
$1^2 + 3^2 = 10$
$1^2 + 0^2 = 1$
$1^2 = 1$
The final result is 1.

Whenever the final result of this procedure is 1, the original number is called a *happy* number. So 23 is a happy number.

21. There are 17 two-digit happy numbers. Try to find all of them. It will save you time and help you look for patterns if you keep a neat record of the above process for each number.

22. Describe any patterns you notice.

LESSON 4.6

In the Lab

You will need:

graph paper

A MYSTERY LIQUID

Reg, Bea, and Gabe were doing an experiment in science class. They had an unknown liquid whose volume they measured in a graduated cylinder. A graduated cylinder is a tall, narrow container that is used for measuring liquid volume accurately. They used a cylinder that weighed 50 grams and measured volume in milliliters. They used a balance to find the weight of the liquid to the nearest gram.

Reg's Data

Volume	Weight
10 ml	16 g
20 ml	32 g
50 ml	80 g
80 ml	128 g

1. Plot Reg's data, with *weight* on the vertical axis and *volume* on the horizontal axis.

2. Does it make sense to connect the points on your graph? Explain.

3. Find an equation relating weight to volume.

4. Estimate the weight of:
a. 60 ml of liquid;
b. 1 ml of liquid.

5. If you add 30 ml to the volume, how much are you adding to the weight? See if you get the same answer in two different cases.

6. If you double the volume, do you double the weight?

Bea's Data

Weight	Volume
16 g	10 ml
32 g	20 ml
48 g	30 ml
64 g	40 ml

7. Plot Bea's data with *volume* on the vertical axis and *weight* on the horizontal axis.

8. Connect the points on your graph with a line and write an equation for the line.

9. Estimate the volume of:
a. 100 g of liquid; b. 1 g of liquid.

10. Compare Bea's graph with Reg's graph. Explain the similarities and differences.

We say that Reg graphed weight versus volume, while Bea graphed volume versus weight.

11. If you add 10 ml to the volume, how much are you adding to the weight? See if you get the same answer in three different cases. Is the answer consistent with what you found in Reg's data?

> **Definition:** *Density* equals weight per unit of volume. This means that to find the density of the mystery liquid, you would find the weight of 1 ml of the liquid. (Actually, scientists use *mass* rather than *weight,* but we will use weight which is equivalent for our purposes.)

12. Find the density of the mystery liquid, using three different pairs of weight/volume values from Reg's and Bea's data. Do all your answers agree? Explain.

13. In problems 4b and 9b, you have found the weight in grams of one ml of liquid, and the volume in ml of one gram. Multiply the two numbers. Explain the result.

THE MYSTERY GROWS

Gabe's Data:

Volume	Weight
10 ml	66 g
20 ml	82 g
40 ml	114 g
60 ml	146 g

14. Draw a pair of axes and label the vertical axis *weight* and the horizontal axis *volume.* Plot Gabe's data.

15. If you double the volume, does the weight double? Check this in two cases.

16. If you add 20 ml, how much weight are you adding? Is this consistent with what you learned from Reg's and Bea's data?

17. *According to Gabe's graph,* what is the weight of 0 ml of the liquid? Does this make sense?

18. What might be the real meaning of the y-intercept on Gabe's graph? Did Gabe make a mistake? Explain.

19. Find the density of the mystery liquid by dividing weight by volume for three different pairs of values from Gabe's data. Do all your answers agree? Explain.

20. Write an equation that expresses weight as a function of volume for Gabe's data.

DIRECT VARIATION

Definition: If the relationship between two variables x and y can be expressed in the form $y = mx$, we call this a *direct variation,* or say that *y varies directly with x.*

21. Which of Reg's, Bea's, and Gabe's data are an example of a direct variation? Explain.

22. Compare Gabe's graph to Reg's. How are they the same and how are they different?

23. There are number patterns in all the data.

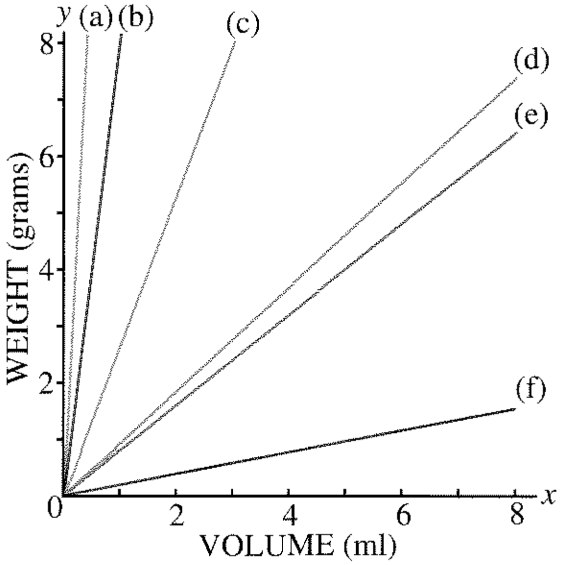
a. What pattern is there in all of Reg's, Bea's, and Gabe's data?

b. What patterns are true only of Reg's and Bea's data?

24. **Summary** What do you know about direct variation? Be sure to discuss equation, graph, and number patterns. You may get ideas from this lesson and Lesson 5.

OTHER SUBSTANCES

25. The graph shows the relationship between weight and volume for some familiar substances. The substances are aluminum, cork, gold, ice, iron, and oak. Which substance do you think is represented by each line? Explain why you think so.



26. Using the graph, estimate the densities of the substances in problem 25.

27. Project

a. Look up the densities of those substances in a science book, almanac, or other reference book. How close were your estimates?

b. Based on your research, what do you think the mystery liquid is? Could it be water? Explain.

LESSON 4.7

Real Numbers and Estimation

You will need:

- graph paper
- graphing calculator (optional)

MEASUREMENT ERROR

The three tables in Lesson 6 contained data that were invented. You can tell because all the points lie exactly on a line. In real experiments measurements can never be exact. This table contains more realistic data.

Volume	Weight
10 ml	32 g
20 ml	63 g
50 ml	146 g
80 ml	245 g

1. Draw and label a pair of axes and plot these points.

2. You cannot draw a straight line through all the points, but draw one that passes as closely as possible to all of them. Be sure your line goes through the origin. (Explain why it must pass through the origin.)

3. What is the equation of the line you drew? (Hint: Choose a point on the line to help you figure this out.)

4. Based on your answer to problem 3, what would you estimate the density of the substance to be?

5. Find the ratio of weight to volume for each data point in the table.

6. Based on your calculations in problem 5, what do you estimate the density of the substance to be?

7. **Summary** You estimated the density of this substance in two different ways. If you did not get the same answer using both methods, explain any differences. Which method do you like better, and why?

ESTIMATING TEMPERATURE

In Chapter 3, Lesson 8, you learned this rule for converting Celsius to Fahrenheit: *Multiply the Celsius temperature by 1.8. Add 32 to the result.*
If

F = the Fahrenheit temperature and
C = the Celsius temperature,

then this statement can be written as a function:

$$F = 1.8\,C + 32.$$

8. Draw and label a pair of axes with F on the y-axis and C on the x-axis. Make a table of values, using values of C from -10 to 30. Use your table to graph the function $F = 1.8\,C + 32$. Label a few points on your graph.

Abe doesn't like to multiply by 1.8. Since 1.8 is a little less than 2, and 32 is a little more than 30, he made up this rule for estimating: *To estimate the Fahrenheit temperature, multiply the Celsius temperature by 2 and add 30.*

9. Using the letters C and F as was done in problem 8, write a function for Abe's rule.

10. Make a table using values of C from -10 to 30 for the function you wrote for Abe's rule. Use your table to graph the function on the same pair of axes as you used in problem 8.

11. Compare the two graphs.

a. How far off would Abe's estimate be if the Celsius temperature were 0?

b. Compare the result from Abe's estimation method with the exact values for several other temperatures. Be sure to try some negative Celsius temperatures. Do you think Abe's method is a good one? Why or why not?

c. There is one temperature for which Abe's estimation method gives the exact value. What is it?

12. For what range of temperatures would you judge Abe's method to be acceptable? Explain.

Sometimes exact answers are important. In everyday life, estimates or *rules of thumb* are often just as good. For example, Mr. and Mrs. Gral, who are planning a trip to Europe, are not really interested in knowing how to make exact temperature conversions. They just want some advice about what to wear.

13. Bea and Abe are making a chart for their parents' reference. Complete it.

Celsius temperature between __ and __	**You should wear:**
	your coolest clothes
	a sweater
	a coat
	heavy coat, gloves, hat, and scarf
	a space suit

ESTIMATING A TIP

Here is a method to figure out how much tip to leave for the server at a restaurant. Say the bill was for $20.73.

- Round up to the next even whole number of dollars, in this case 22.
- Add half of the number you got to the number, in this case 22 + 11 = 33.
- Round up to the next multiple of five, in this case 35.
- Divide by ten to get the tip, in this case $3.50.

14. What percentage of $20.73 is $3.50? (Round off your answer.)

15. Does this method always give the same percentage of the bill? Try it for several amounts to see whether the percentage varies. If it does, what seem to be the lowest and the highest value it will give?

Here is another method to figure out the tip.

- Divide the amount of the bill by ten. (In this case you would get $2.07.)
- Multiply the result by two. (In this case you would get $4.14.)
- Take the average of the two numbers, rounded to the nearest nickel.

16. a. What is the tip by this calculation?

b. What percentage of the bill is it?

17. Does the second method always give the same percentage of the bill? Explain.

18. **Summary** Compare the two methods. Explain which one you prefer and why.

19. What percentage of the bill do you think is an appropriate tip? Create your own method to figure it without a calculator.

LESSON 4.8

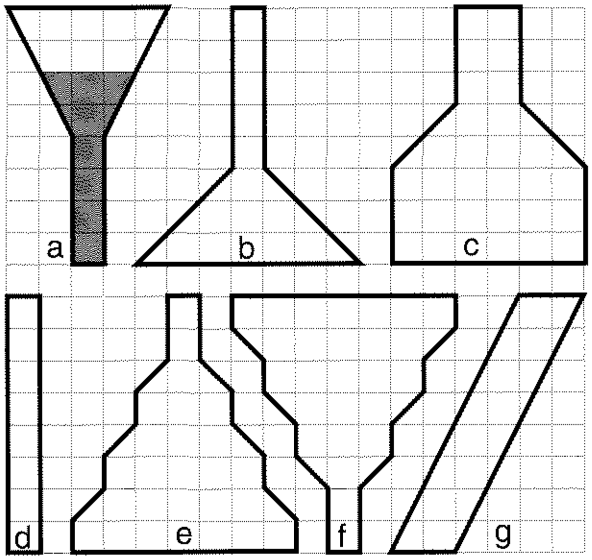
Jarring Discoveries

You will need:

- centimeter graph paper
- jar lids
- string
- centimeter rulers

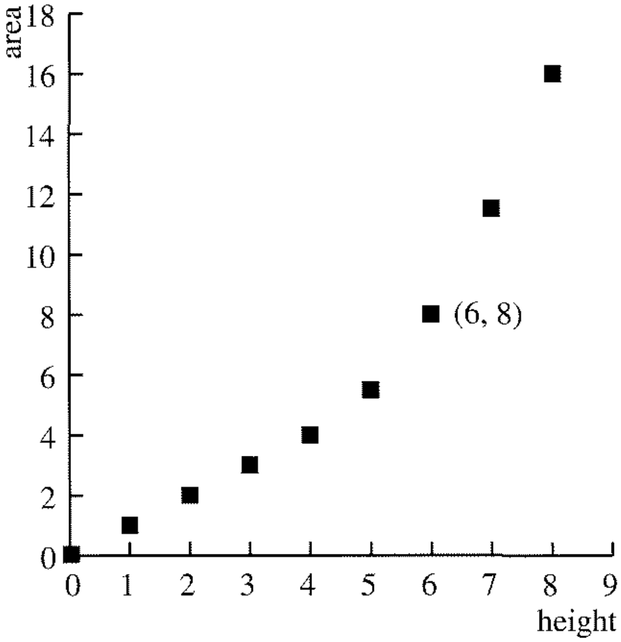
FLAT SCIENCE

Doctor Dimension is a flat scientist. He stores two-dimensional liquids in two-dimensional jars, like the ones shown in this figure.



One day, as part of his scientific research, he decides to graph the amount of liquid in a jar as a function of the height of liquid. Since he lives in a two-dimensional world, liquid is measured in square units. For example, jar (a) is filled to a height of six units and contains eight square units of liquid.

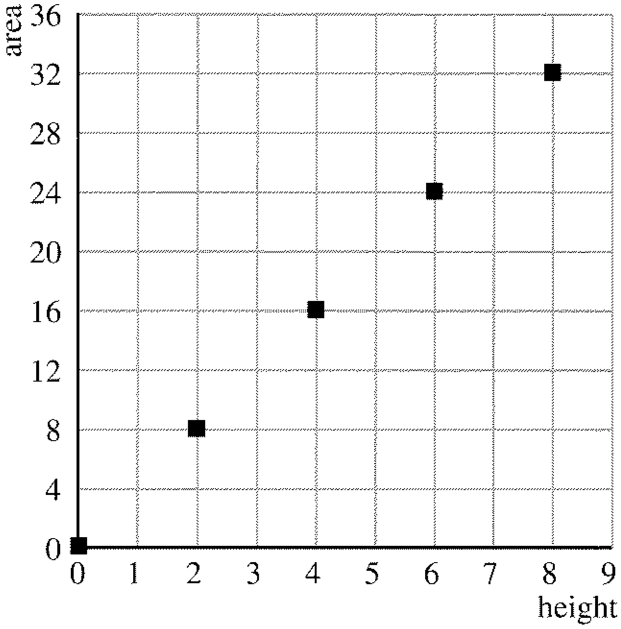
The following graph represents jar (a).



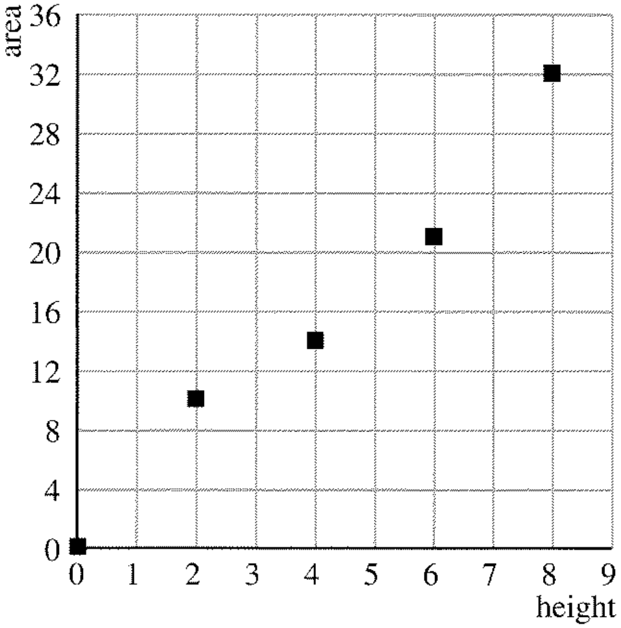
1. Some of the dots lie on one straight line. In which part of the graph does this happen? Explain why this is so.
2. Make a graph for each of the remaining jars.
3. For which jars is the area of liquid a direct variation function of the height? Explain.

4. Draw two different jars for each graph below.

a

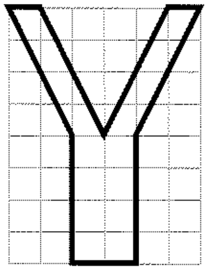


b



5. Draw a jar for part (a) of problem 4 for which the area of liquid is *not* a direct variation function of the height.

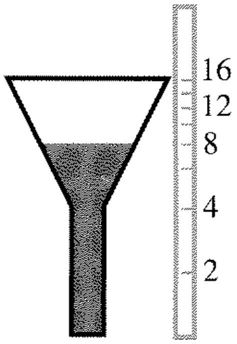
6. Predict the shape of the graph for this jar. Then test your prediction.



7. **Summary** Explain how the shape of the jar affects the shape of the graph. Explain what it takes for a jar to have a graph that is a straight line through the origin.

8. **Generalization** How do you think the shape of a three-dimensional jar affects the shape of the graph of the volume of liquid as a function of height? What jar shapes correspond to a direct variation function?

A dipstick can be used to measure the amount of liquid in a jar, but the dipstick must be specially designed for the jar. For example, the following dipstick would work for jar (a).



On it, area is marked off with a tick for every two square units.

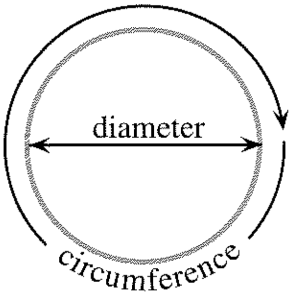
9. Note that the dipstick ticks are not evenly spaced. Explain why.

10. Which jars would have a dipstick whose ticks are evenly spaced? Explain.

11. **Project** Draw an accurate dipstick for each of several different jars. Write a report showing sketches of the jars and their dipsticks, and explain your method.

JAR LIDS: CIRCUMFERENCE

For this section, use jar lids of at least five different sizes, including one very small one and one very large one.



12. Measure the diameter and circumference of each of the jar lids in centimeters, as accurately as possible. (Use the string to help find the circumference.) Make a table showing your data.

13. Make a graph of your data, putting diameter on the x-axis and circumference on the y-axis. Don't forget to include a point for a lid having diameter 0.

14. What is the relationship of circumference to diameter for each jar lid? Describe it in words and with an equation. Explain how you figured it out.

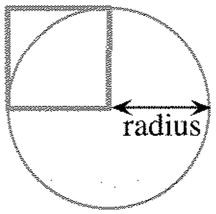
15. Is the relationship between diameter and circumference an example of direct variation? Explain.

16. According to your data, what is the approximate value of the ratio of circumference to diameter?

JAR LIDS: AREA

17. Estimate the area of the top of each jar lid by tracing around it on centimeter graph paper and estimating the number of square centimeters it covers. Make a table and a graph of the relationship between diameter and area, including a point for a lid having diameter 0.

18. Is the relationship between diameter and area an example of direct variation? Explain.



The figure shows a square whose side equals the radius of the circle.

19. For each jar lid, calculate the area of a square like the one shown in the figure. Add a column for these data in your jar-lid area table.

20. Graph the area of the circles as a function of the area of the squares.

21. What is the relationship between the area of the circles and the area of the squares? Describe it in words and with an equation. Explain how you figured it out.

22. Is the relationship between the area of the circles and the area of the squares an example of direct variation? Explain.

23. According to your data, what is the approximate value of the ratio of the area of the circle to the area of the square?

24. **Summary** According to your data, what is the relationship between the area of a circle and its radius? The area of a circle and its diameter? Explain.

REVIEW *DIVIDING ON A CALCULATOR*

Phil used his calculator to find the reciprocal of 7, and got the number 0.1428571429. Lyn's calculator, on the other hand, gave the number 0.1428571428.

25. Explain how two calculators can give different results, even though neither is defective.

Phil's grandfather does not believe in calculators. He said, "Do you really believe either number is the reciprocal of 7? I have news for you. Multiply each one by 7 without a calculator, and you'll see why you should not trust these machines."

26. Work with a classmate. Do the two multiplications on paper to see who was right, Phil, Lyn, or their grandfather. Explain your results.

The grandfather added, "To find the real reciprocal of 7, you have to use good old-fashioned long division."

27. Find the real reciprocal of 7.

28. **Report** Write a letter to Lyn and Phil's grandfather, explaining why students are allowed to use calculators nowadays. Your letter should include, but not be limited to:

- Answers to the grandfather's probable objections;
- A table showing the real reciprocals of the whole numbers from 0 to 10, and the reciprocals as given by Lyn's and Phil's calculators;
- An explanation of how you can find the real reciprocal by using a calculator;
- An argument explaining why Lyn's or Phil's calculator is the better one for the purpose of finding reciprocals.

29. Make a *division table* like this one. Extend it to show whole-number numerators and denominators from 0 to 10. You may use a calculator, but enter only exact answers. Look for patterns and work with a partner. Some answers were entered for you.

Numerators

		0	1	2	3	...
Denominators	0					
	1					
	2	0	0.5	1	1.5	...
	3					
	...					...

30. What patterns do you notice about the row of your table for denominator 7?

31. Learn how to use the FIX mode on your calculator.

THINKING
WRITING

4.B Direct Variation

You will need:

graph paper

POINTS ON LINES

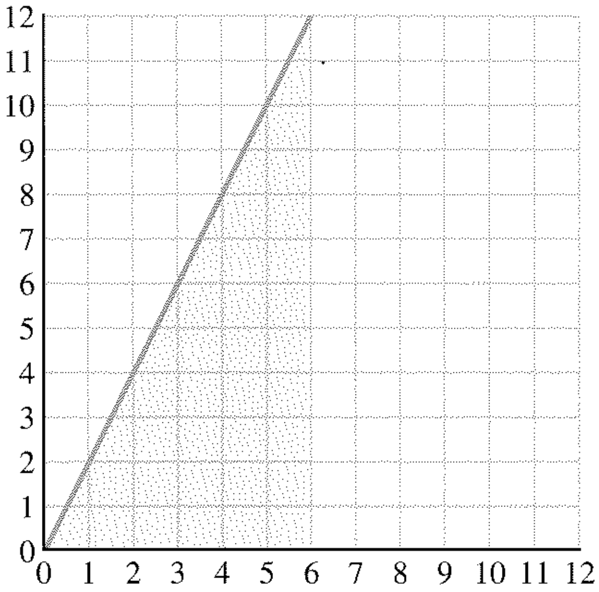
1. Choose a number m, and draw the graph of the equation $y = mx$. Choose any point (a, b) on the line.
   a. Is the point $(2a, 2b)$ on the line?
   b. Is the point $(3a, 3b)$ on the line?
   c. Is the point (ka, kb) on the line for any value of k?

2. Refer to the line you drew in problem 1.
   a. Is the point $(a + 1, b + 1)$ on it?
   b. Is the point $(a + k, b + k)$ on the line for any value of k?

3. **Report** Repeat problems 1 and 2 for several graphs of the form $y = mx$, $y = x + b$, and $y = mx + b$. If a point (a, b) is on the line, in what case is (ka, kb) on the line? What about $(a + k, b + k)$?

AREA FUNCTIONS

4. The graph shows $y = 2x$. The region between the line and the x-axis from $x = 0$ to $x = 6$ is shaded.
   a. What is the area of the shaded region?
   b. What is the area of the region between the line and the x-axis from $x = 0$ to $x = 4$?



Endpoint	Area
$x = 1$	
$x = 2$	
$x = 3$	
$x = 5$	
$x = a$	

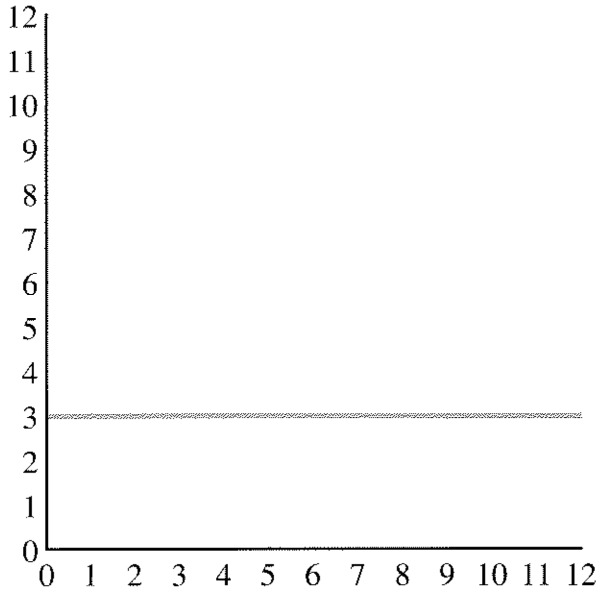
5. Copy and complete the table giving the area between the line and the x-axis from $x = 0$ to the given endpoint value of x.

6. Find a function relating the area to the endpoint value of x.

7. Is the area function you wrote an example of direct variation? Explain.

Endpoint	Area
$x = 1$	
$x = 2$	
$x = 3$	
$x = 5$	
$x = a$	

8. The graph shows the line $y = 3$. Copy and complete the table giving the area between the line and the x-axis from $x = 0$ to the given endpoint value of x.



9. Find a function relating the area to the endpoint value of x.

10. Is the area function you wrote an example of direct variation? Explain.

11. **Report** Repeat problems 4 through 7 for several other lines. For which lines did you find area functions that are examples of direct variation? What generalizations can you make? Write an illustrated report about your results.

LESSON

4.9

Rules of the Road

You will need:

graph paper

UNIT CONVERSION

When you talk about the speed you are traveling, you usually give the speed in *miles per hour.* In this lesson it will be useful to give speed in *feet per second.* Use the fact that 100 miles per hour (mph) is about 147 feet per second (fps).

1. How many fps is 50 mph?

2. Complete the table to show the relationship of miles per hour to feet per second. Extend the table up to 80 miles per hour.

mph	fps
10	14.7
20	

3. If you made a graph from your table with *mph* on the x-axis and *fps* on the y-axis, what would the graph look like? (If you are not sure, draw it.)

4. If you were traveling at 1 mph, how fast would you be going in fps?

STOPPING DISTANCE

To stop a car in an emergency, you first react and then put on the brakes.

stopping distance =
reaction distance + braking distance

5. What kinds of things do you think would affect reaction time and distance? Braking time and distance?

Reaction time is often considered to be about 3/4 of a second, but how far you travel during this time depends on how fast you are going.

6. Reaction distance:

a. Figure out how many feet you would travel in 3/4 of a second if you were going at various speeds (10 mph, 20 mph, etc.). Make a table to display your data.

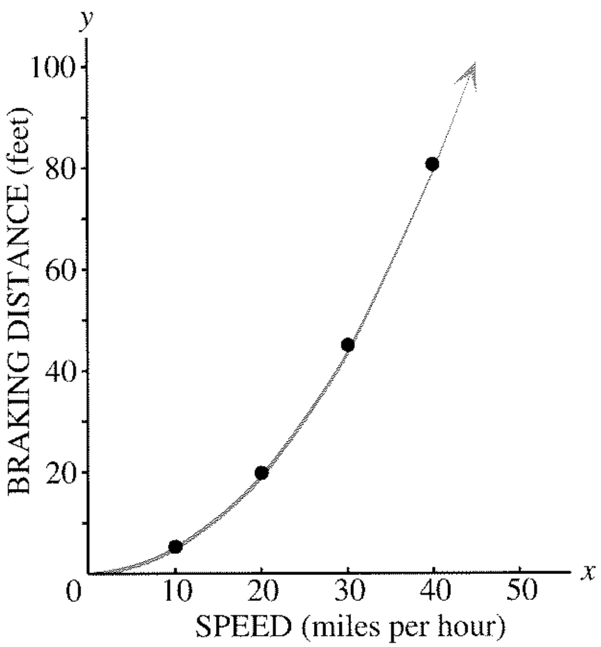
b. Graph your data. Put *reaction distance in feet* on the y-axis and *speed in miles per hour* on the x-axis.

c. Describe the relationship between the two variables on your graph.

7. Braking distance: A formula for finding braking distance in feet is to take the speed in miles per hour, square it, and divide the result by 20. For example, if the speed were 10 mph, the braking distance would be $(10)^2/20 = 100/20 = 5$ feet.

a. The graph on the next page shows the relationship between the braking distance (in feet) and the speed (in miles per hour). All the points on the graph were found by using the formula. Make a table showing the coordinates of at least five points on the graph.

b. According to the table and graph, if you double your speed, will you double your braking distance? Explain, giving examples.



8. Total stopping distance: Use your tables and graphs from problems 6 and 7 to make a table with the headings shown. Use at least five different speeds.

Speed (mph)	Reaction distance (feet)	Braking distance (feet)	Total stopping distance (feet)

SAFE DISTANCE

It is estimated that about 30 percent of all automobile accidents are caused by following too closely. Two rules of thumb for avoiding accidents follow.

Rule 1: The 3-Second Rule. Notice when the vehicle in front of you passes some object, such as a road sign. Then time approximately three seconds by counting, "One-thousand-one, one-thousand-two, one-thousand-three." If you pass the same object before you get to one-thousand-three, you are following too closely.

Rule 2: The 1-for-10 Rule. Leave one car length between you and the car in front of you for every 10 mph of driving speed.

9. **Exploration** Which rule do you think is safer? Taking into account what you found out about stopping distance, what do you think would make a good rule of thumb?

To compare the two rules, it helps to convert miles per hour to feet per second, so that all units are in feet and seconds.

10. a. Copy and complete the table to show the distance traveled in three seconds at the speeds given. Extend the table up to 100 miles per hour.

Speed (mph)	Speed (fps)	Distance (ft)
10	14.7	44.1
20		

b. According to the table, how many feet would a car traveling at 50 mph cover in three seconds?

c. If you were instructed to stay three seconds behind the car in front of you, how many feet would that be, if you were traveling at 70 mph?

d. If you slowed down to 35 mph, could you cut your following distance in half? Explain.

e. If you drew a graph with speed on the y-axis and *distance traveled in three seconds* on the x-axis, what would it look like? Explain. If you are not sure, sketch the graph.

11. Most cars are about 14 to 18 feet in length. Choose a car length in this interval and make a table showing safe following distances at certain speeds according to Rule 2.

Speed (mph)	**Speed (fps)**	**Safe distance (car lengths)**	**Safe distance (feet)**
10	14.7	1	
20		2	

12. Use your tables to compare Rule 1 and Rule 2. How are they different? Which one suggests greater caution? Explain.

13. Should one evaluate Rule 2 based on its implementation using a small-car length or a large-car length? Explain.

14. **Report** Use the information about total stopping distance to decide whether you agree with the advice given by Rule 1 or by Rule 2, or whether you would suggest a different rule. Write a paragraph explaining your opinion.

DISCOVERY ROUNDING

Because of measurement error, it is meaningless to say that someone weighs 157.2490368 pounds. No scale is that accurate, and even if it were, one does not need that level of accuracy. For most purposes, it is satisfactory to talk of someone's weight to the nearest pound, so this number should be rounded off to 157.

When dealing with amounts of money, one usually rounds off to the nearest cent. In some cases, one rounds up, or down. When doing work with *real* numbers, make sure you do not copy answers from your calculator without thinking of whether you should round off, round up, or round down.

15. If you buy one 95-cent pastry at the Columbia Street Bakery, you will be charged $1.00 even. But if you buy two pastries, you will be charged $2.01.

a. What is the sales tax in this town?

b. Does the cash register round off to the nearest cent, or does it round up or down? Explain.

16. In the same town, if you buy a 94-cent soda at Eddie's, you will be charged $1.00. If you buy two sodas, you will be charged $1.99. Does this cash register round off to the nearest cent? Does it round up or down? Explain.

LESSON 4.10

Up in the Air

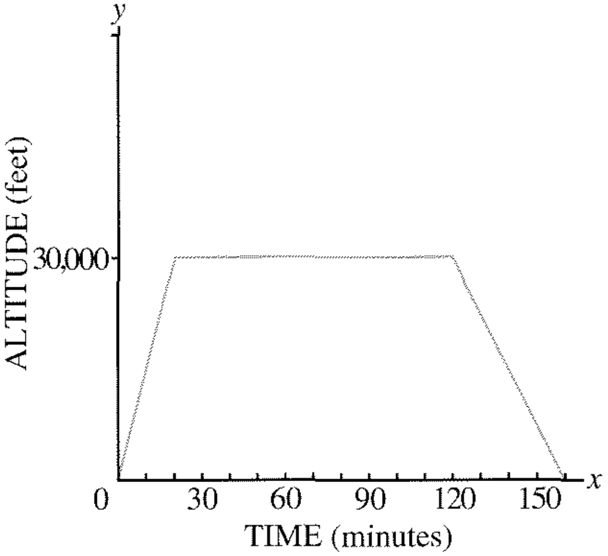
You will need:

graph paper

People rarely travel at constant speeds. Almost all travel involves speeding up and slowing down. However, sometimes to simplify a problem it is useful to use the average speed over a given period of time. In this lesson we will use the average speed.

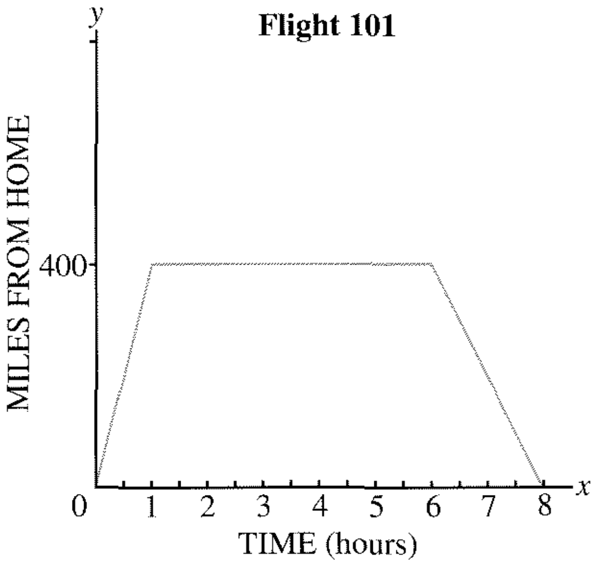
MULTIPLE MEANINGS

The graph below shows the relationship between the altitude of the airplane and the time after take-off.



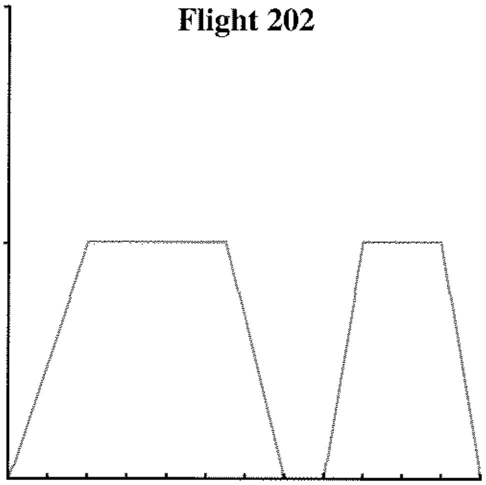
1. How high was the airplane 20 minutes after take-off?
2. How long after take-off did the airplane reach its cruising altitude?
3. How long did the plane cruise at a constant altitude before descending?
4. Can you figure out the speed of the airplane from this graph? Explain.

The graph below shows that Flight 101 left its home airport at 8 A.M. and flew to the town of Alaberg. It stayed in Alaberg for several hours and then returned to its home airport.



5. According to the graph, how far away is Alaberg?
6. How long did it take Flight 101 to get to Alaberg?
7. How long did the plane stay in Alaberg?
8. Can you figure out the speed of the airplane from this graph? Explain.

Someone made this graph about Flight 202, but accidentally left off the labels and the scale for the axes.

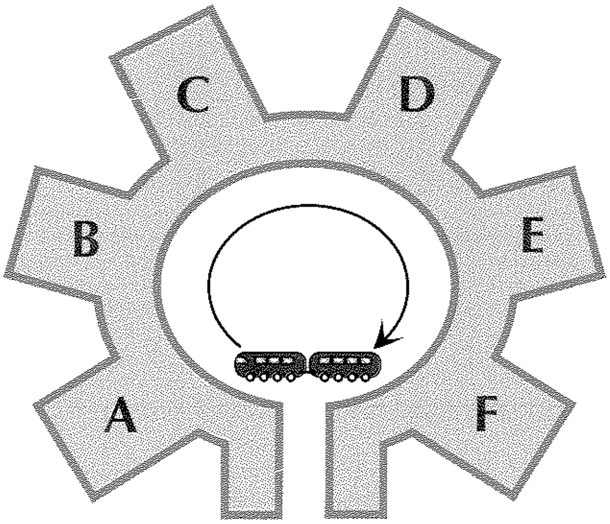


9. Copy the graph and label the axes like those in the figure just before problem 1. Write a description of what the graph conveys.

10. Make another copy of the graph and label the axes like those in the figure preceding problem 5. Write a description of what that graph conveys.

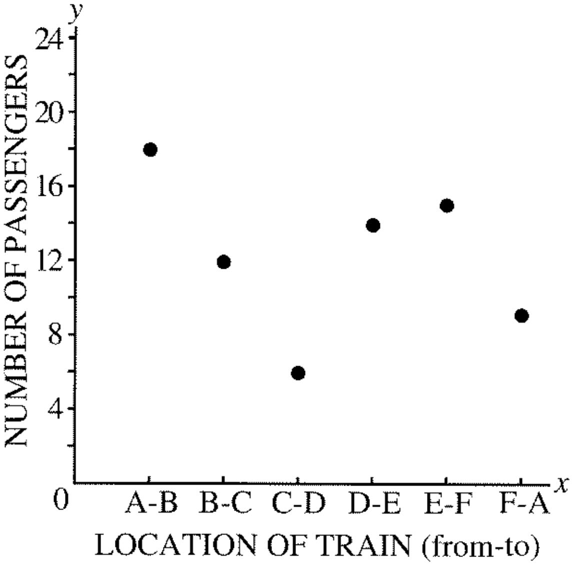
11. What else might the axes and scale be for the graph about Flight 202? Make up another possibility and write a description of what your graph shows.

DISCRETE AND CONTINUOUS GRAPHS



Alaberg has a large airport with several terminals. A small train runs through the airport, carrying passengers between the terminals. Passengers use this train when they have to transfer from one plane to another. The graph shows the relationship between the location of the train between the terminals and the number of passengers in the train.

12. Write a description of what is conveyed by this graph.



13. Can you tell how many passengers got on and off at each terminal? Explain.

14. Can you tell if the train was ever empty?

15. Can you tell from this graph how fast the train was traveling?

The Alaberg Airport Express is a van service that carries passengers between the city and Alaberg Airport. A group of math teachers is holding a convention in Alaberg, and 1024 people have arrived at the airport. They all need to get into the city.

16. If the Alaberg Airport Express van holds 20 people, how many trips will be needed to take all the people into the city?

17. If more vans were available, fewer trips would be needed per van. If 15 vans were available, and the trips were divided as evenly as possible among the vans, what would be the maximum number of trips that any van would need to take?

18. Copy and complete the table to show the relationship between vans available and maximum number of trips per van necessary. (Once again, assume that the trips would be divided as evenly as possible among the vans.)

Number of vans	Max number of trips per van necessary
1 . . . 15	

19. a. Make a graph from your table.

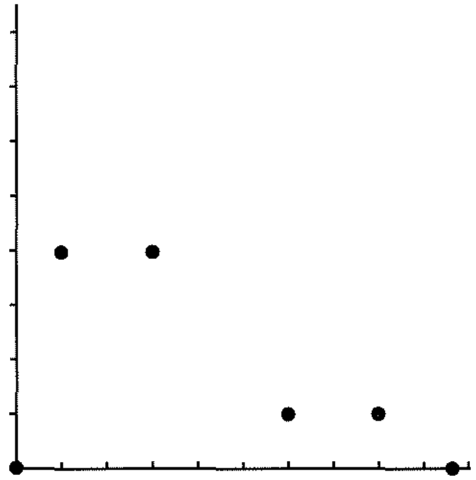
b. What is the rule for finding the maximum number of trips per van necessary, given the number of vans?

20. The graphs you used in problems 12 and 19 involved points instead of lines. Explain why it does not make sense to connect these points.

Definition: If the points are not connected on a graph, it is called *discrete*. If the points are connected, it is called *continuous*.

YOUR OWN GRAPHS

21. The meaning of this graph is still up in the air until you add some things to it. Copy the graph, label the axes, and show the scale. If it makes sense, connect the points. Tell what the graph conveys.



22. Make up a discrete graph. Label the axes and indicate the scale. Write a description of what the graph conveys.

23. Repeat problem 22 for a continuous graph.

LESSON 4.11

Horizontal and Vertical Lines

You will need:

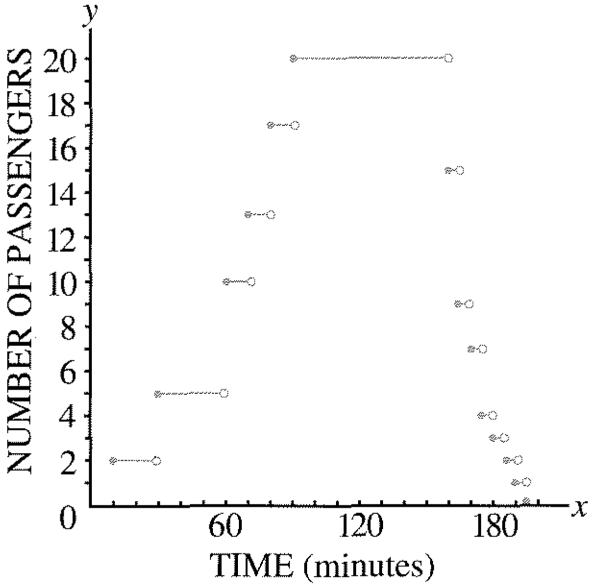
graph paper

STEP FUNCTIONS

1. This graph shows how the number of passengers in the Alaberg Airport Express van changes over time. The graph shows a trip between the city and the airport.

a. Write a description of what is shown by this graph.

b. Why is there a long horizontal line on the graph?



Definitions: This graph is an example of a *step function.* Note that the endpoints of the steps are either filled-in (this is called a *closed circle*), or hollow (this is called an *open circle*).

For a given value of x, the value of y can be found by looking for the corresponding point on the graph. Only points on the steps and closed circles are considered to be on the graph. The open circles are there to show where the step ends, but that point is not considered to be on the graph. This way a given x-value has only one corresponding y-value.

2. After 60 minutes, how many people were in the van?

HORIZONTAL AND VERTICAL LINES

3. a. Graph the vertical line through the point $(1, -2)$.

b. Label four more points on this line.

c. Which coordinate is the same for all the points on the line, the x-coordinate or the y-coordinate?

4. a. Graph the horizontal line through the point $(1, -2)$.

b. Label four more points on this line.

c. Which coordinate is the same for all the points on the line, the x-coordinate or the y-coordinate?

5. a. The equation of a line is $y = -3$. There is no x in the equation because the value of y does not depend on the value of x. Graph this equation.

b. Did you graph a horizontal or a vertical line?

6. a. The equation of a line is $x = 6$. There is no y in the equation because the value of x does not depend on the value of y. Graph this equation.

b. Did you graph a horizontal or a vertical line?

7. a. Graph the vertical line through (2, -5). Write its equation.

b. Find the coordinates of any point on the line.

c. Find the coordinates of any point to the right of the line.

d. Find the coordinates of any point to the left of the line.

e. For each part (b), (c), and (d), answer this question: What do you think all the points chosen by students in your class have in common?

8. The equation of a line is $y = 5$. If possible, answer these questions without graphing the line.

a. Is the line vertical or horizontal?

b. Where does the point (4, -2) lie in relation to the line? Explain.

c. Write the coordinates of one point on the line and one point not on the line.

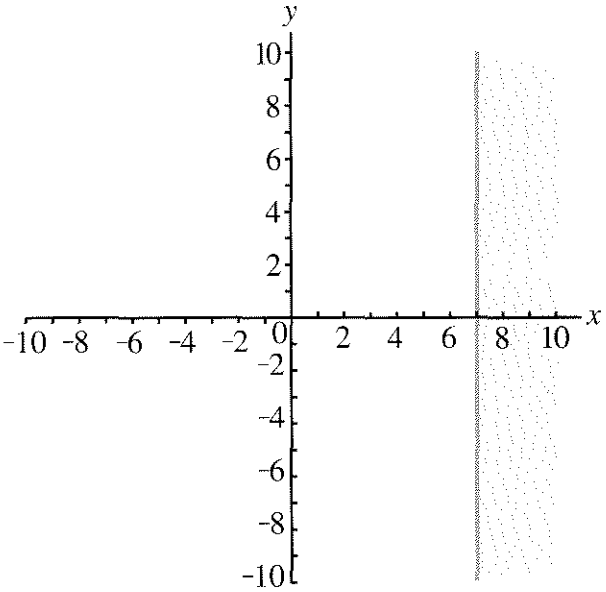
d. What can you say about the y-coordinate of any point that lies on the line? Below the line? Above the line?

INEQUALITIES

The x-coordinate of any point that lies to the left of the vertical line $x = 6$ must be a number less than 6. For example (2, 7) is such a point, since $2 < 6$. The expressions $2 < 6$ and $x < 4$ are examples of *inequalities.*

9. The mathematical shorthand for *less than* is <. What are the mathematical symbols for *greater than, less than or equal to,* and *greater than or equal to*?

Inequalities can be used to describe sets of points on a graph. For example, all the points that lie on or to the right of the line $x = 7$ can be described by the inequality $x \geq 7$.



10. Graph each set of points given. Use one or more inequalities to describe it.

a. All points that lie on or below the line $y = -1$

b. All points that lie on or above the x-axis

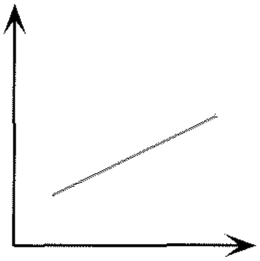
c. All points that lie on or between the vertical lines $x = 3$ and $x = 6$

11. **Report** Write an illustrated report on horizontal lines, vertical lines, and inequalities.

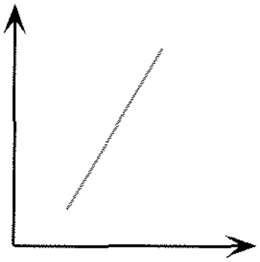
DISTANCE VS. TIME, AGAIN

These graphs represent the motion of Paul's car. The vertical axis shows distance from his house, and the horizontal axis shows time.

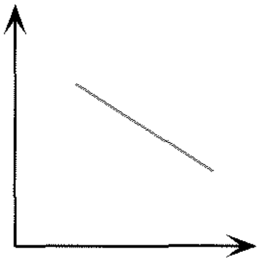
a.



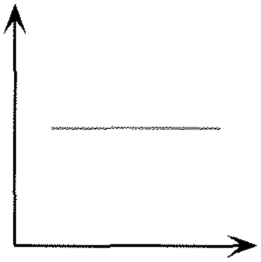
b.



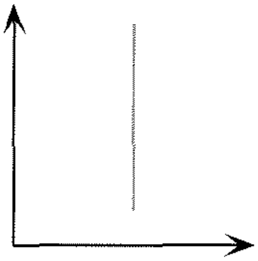
c.



d.



e.



12. Describe the trips shown in each graph. Are all of them possible?

COST OF MAILING A LETTER

In January of 1991 the United States Postal Service raised its rates for first-class mail. It printed the following table in a flyer for postal customers.

13. Answer questions (a-c) using the information in the following table.

Single-Piece Letter Rates:

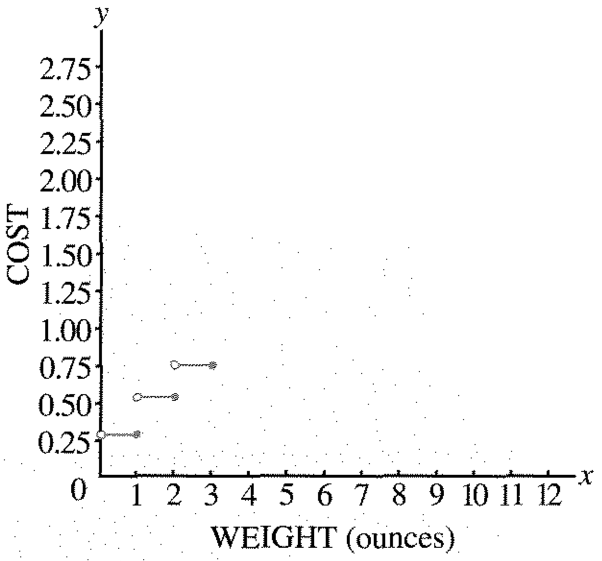
Pieces not exceeding (oz)	The rate is	Pieces not exceeding (oz)	The rate is
1	$0.29	7	$1.67
2	$0.52	8	$1.90
3	$0.75	9	$2.13
4	$0.98	10	$2.36
5	$1.21	11	$2.59
6	$1.44		

a. How much does it cost to mail a letter weighing 7 and 1/2 ounces?

b. How much does it cost to mail a letter weighing exactly 3 ounces?

c. Would it be possible for a letter to cost 45 cents to mail? If so, how much would it weigh? If not, explain why not.

14. Use the data in the table to graph the relationship of cost to weight. It is a step function. Copy and complete this graph.



15. Study the table. What is the rule being used to determine these rates?

LESSON

4.12 Complicated Areas

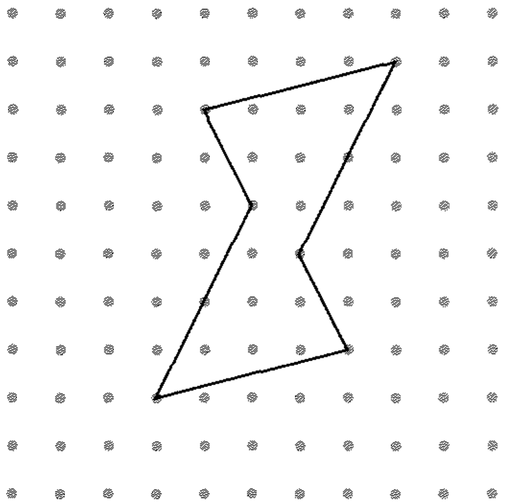
You will need:

geoboards

dot paper

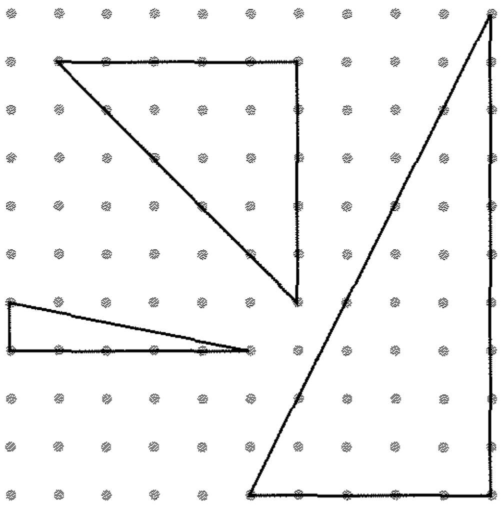
DISCOVERING AN AREA FORMULA

1. a. Find the area of this figure.
   b. Explain how you did it, with the help of illustrations on dot paper.
   c. Compare your approach with other students' work.



In the figure above, the rubber band is in contact with 8 geoboard pegs (which we will call *boundary dots*). The figure encloses 12 inside pegs, which we will call *inside dots.*

2. For each figure, give the number of boundary dots, the number of inside dots, and the area.



3. **Exploration** Try to figure out the relationship between boundary dots, inside dots, and area. (Hints: Sketch many simple figures, count their dots, and find their areas. Keep detailed and clear records. Start by working on the problem for figures having zero inside dots, then one inside dot, and so on.) Keep records of your work in a table like this one.

Boundary Dots	**Inside Dots**	**Area**
...	...	...

4. Make three figures having 3 boundary dots and 0 inside dots. Find the area of each figure.

5. Make three figures having 4 boundary dots and 0 inside dots. Find the area of each figure.

6. Make three figures having 5 boundary dots and 0 inside dots. Find the area of each figure.

7.
  a. If two figures have no inside dots and the same number of boundary dots, what can you say about their areas?
  b. What happens to the area if the number of boundary dots increases by 1?

8. a. Predict the area of a figure having 10 boundary dots and 0 inside dots.
  b. Check your prediction by making three such shapes and finding their areas.
  c. What would the area of a figure having 99 boundary dots and 0 inside dots be?

9. **Generalization** Explain how one could find the area of a figure having *b* boundary dots and 0 inside dots, without making or drawing the figure.

10. Make figures having 10 boundary dots and 1, 2, 3, etc. inside dots. For each one, find its area. Keep your work organized in a table.

11. What happens to the area when the number of inside dots increases by 1?

12. a. Predict the area of a figure having 10 boundary dots and 10 inside dots.
  b. Check your prediction by making three such shapes and finding their areas.
  c. What would the area of a figure having 99 boundary dots and 101 inside dots be?

13. **Generalization** Explain how one could find the area of a shape having *b* boundary dots and *i* inside dots, without making or drawing the figure. You have discovered *Pick's Formula.*

14. Use the result from problem 13 to check your answers to problems 1 and 2.

DISCOVERY *PATTERNS AND FUNCTIONS*

15. Find as many functions of x as possible whose value is 5 when x is 2.

16. Multiply.

$$\left(1-\frac{1}{7}\right)\cdot\left(1-\frac{2}{7}\right)\cdot\left(1-\frac{3}{7}\right)\cdot\ldots\cdot\left(1-\frac{9}{7}\right)$$

17. 1
1+2+1
1+2+3+2+1
1+2+3+4+3+2+1
What do you notice about these sums? Explain the pattern, using a figure if you can.

THINKING WRITING

4.C Letter Strings

In abstract algebra, letters do not stand for numbers. Abstract algebra has many applications, for example, to particle physics or to the analysis of the Rubik's cube. Here is a simple example.

THE YZ GAME

In this game, starting with a string of Y's and Z's, the object is to simplify the string by following strict rules. The rules are:

YYY can be erased.
ZZ can be erased.
the commutative law: YZ = ZY.
E is the empty string (a string with no Y's or Z's).

Examples:

a. YZZYYZYZYYZ (erase ZZ)
Y YYZYZYYZ (erase YYY)
ZYZYYZ (commute YZ)
ZZYYYZ (erase ZZ and YYY)
Z (can't be simplified)

b. ZYYYZ (erase YYY)
Z Z (erase ZZ)
E (the empty string is left)

1. Simplify the strings.
a. YZYZZYYZ b. YYYYZZYZY
c. YZYZYZYZYZYZYZYZZZYZYZYYZY

Including the empty string E, there are six essentially different strings that cannot be simplified. They are called the *elements of the YZ group.*

2. Find all the elements of the YZ group.

The symbol ↔ represents the operation *put together and simplify.* For example:

Y ↔ YY = E
YZ ↔ YZ = YY
Y ↔ E = Y

3. Compute.
a. E ↔ YZ b. YZ ↔ YY
c. Z ↔ YZ

4. Find the missing term.
a. YZ ↔ ___ = E
b. Z ↔ ___ = YZ
c. YY ↔ ___ = Z

For the YZ group, ↔ works a little bit like multiplication. Another way to write the first two rules is

$$Y^3 = E \text{ and } Z^2 = E.$$

5. The only powers of Y are: Y, Y^2, and E. Explain.

6. Find *all* the powers of each element of the YZ group.

7. Simplify. (Show your work.)
a. Y^{1000} b. $(YZ)^{1001}$

8. Make a ↔ table.

9. What element of the group works like 1 for multiplication?

10. What is the reciprocal of each element? (In other words, for each element, what element can be put together with it to get the 1?)

THE yz GAME

For this group, the rules are:

yyy can be erased.
zz can be erased.
yzy = z.
The empty string is called e.
There is no commutative law.

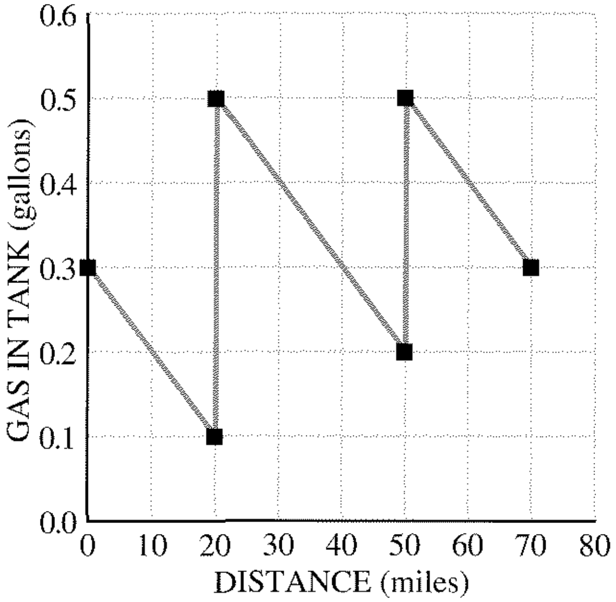
11. 💡 Do problems 1-10 for the yz group. (Hint: zyy and yyz *can* be simplified.)

12. **Report** Write a report on the yz group.

Essential Ideas

FUEL VS. DISTANCE

Gabe's scooter gets good mileage, but it has a small tank. The graph below shows how much gas was in his tank during one trip he took.



1. Write a paragraph describing Gabe's trip. Include the answers to these questions: How much gas did Gabe start with? How much did he end with? How many times did he stop for gas? How much gas did he use for the whole trip? How far did he travel before stopping each time? What is probably the capacity of his gas tank? How many miles did he get per gallon?

2. The gas station stops took ten minutes each. Gabe left home at 9 A.M. and arrived at his destination at 11:05 A.M. How fast does the scooter go?

3. In what ways might this graph be unrealistic?

EQUATIONS AND GRAPHS

4. Make a graph of several (x, y) pairs having the property that the sum of x and y is 16. Connect the points on your graph. Write the equation of your graph.

5. Write the equation of:
   a. a line through the origin containing the point (2, 5);
   b. another first-degree polynomial containing the point (2 ,5);
   c. a second-degree polynomial containing the point (2, 5).

These questions are about the graph of the equation $y = -x^2 + 2$.

6. Which of these points are on it?
   (3, -11) (-3, 11) (3, -7) (-3, -7)

7. The point $(-6, y)$ is on it. What is y?

8. The point $(x, -14)$ is on it. What are the two possible values of x?

For each of the equations below, if possible, find an (x, y) pair for which
   a. x is negative and y is positive;
   b. x is positive and y is negative;
   c. x and y are both negative.

9. $y = 4x$ **10.** $y = x^2 - 2$

11. $y = x(x - 1)$ **12.** $y = -2x + 6$

13. Which of the above four equations' graphs
   a. are straight lines?
   b. pass through the origin?

14. If possible, sketch the graph of a zero-degree, first-degree, second-degree, and third-degree polynomial function which passes through all quadrants but the first.

For problems 15 through 17:

a. Plot the points given in the table.
b. Study the table and your graph. Describe the relationship between the x-value and y-value of each pair.
c. Use the pattern you found to add more points to your table and graph.
d. Write an equation that tells how to get the y-value from the x-value.

15.

x	y
0	1
1	3
-1	-1
3	7

16.

x	y
2	9
3	6
-3	-6
4.5	4
4	4.5

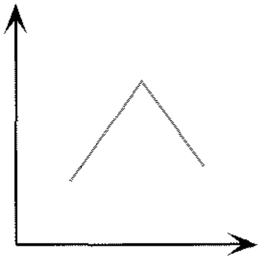
17.

x	y
3	8
-2	3
-1	0
0	-1
1/2	-3/4
2	3

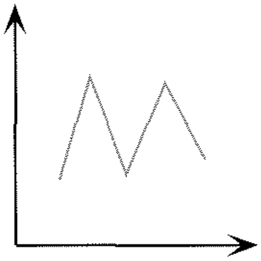
DISTANCE VS. TIME

18. These graphs represent the motion of Paul's car. The vertical axis shows distance from his house, and the horizontal axis shows time. Write a short paragraph describing the trip summarized by each graph.

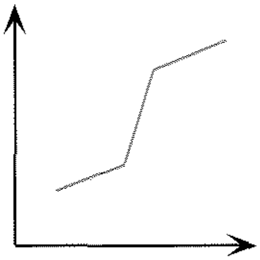
a.



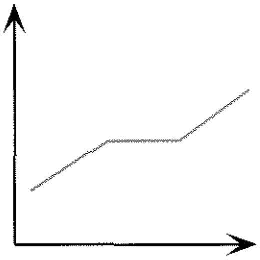
b.



c.



d.



POSTAL PROBLEM

The post office puts size restrictions on first-class mail. Standard letters that are 1/4 inch thick or less must meet these requirements for width and height.

- The height is at least 3 and 1/2 in. and cannot exceed 6 and 1/8 in.
- The width is at least 5 in. and not more than 11 and 1/2 in.

19. The first condition can be written $3.5 \leq \text{height} \leq 6.125$. (This is called a *compound inequality*.) How would you write the second condition?

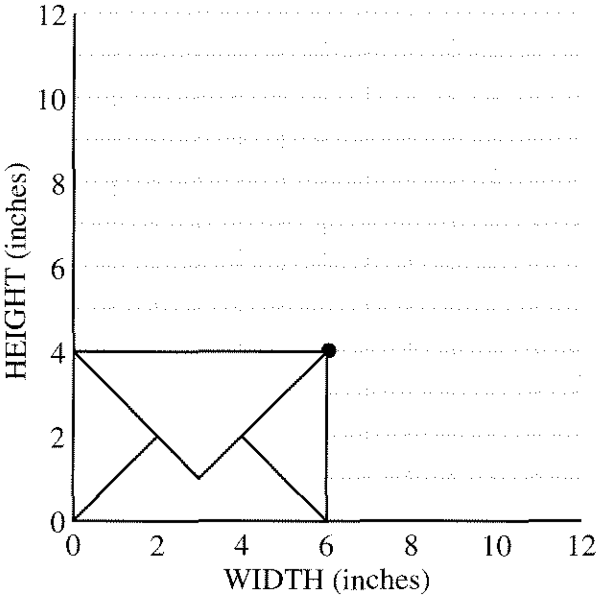
20. Sketch (to scale) and give the width, height, and area of each of these letters.

a. The letter having the least possible area

b. The letter having the greatest possible area

c. The tallest, thinnest letter

d. The shortest, widest letter



You can use a graph to show allowable dimensions of a letter. In the graph above, the point (6, 4) represents the dimensions of a letter that is 4 in. high and 6 in. wide.

21. Plot four points for the four envelopes you listed in problem 20. (Don't draw the envelopes!)

22. Write the equations of two horizontal and two vertical lines through those points.

23. The four points should form a rectangle. Find some points inside the rectangle, outside the rectangle, and on the rectangle. Which points represent allowable dimensions of letters? Explain, using examples.

In order to avoid extra fees, your letter must satisfy the following restriction.

- The width divided by the height must be between 1.3 and 2.5, inclusive.

24. Write a compound inequality for this restriction.

25. Find the ratio of the width to the height of each letter you listed in problem 20. Which ones meet the new requirement?

26. a. Experiment with your calculator until you find an allowable width and height that have a ratio of 1.3. On your graph, plot these dimensions. Draw a line through this point and the origin.

b. Find other points on the line. What is the ratio for each one? Explain.

c. Repeat (a) and (b) for the ratio 2.5.

27. Check the ratio for points between the two lines, above the upper line, and below the lower line.

28. Explain how to use the graph to find

a. dimensions that satisfy all the rules;

b. dimensions that satisfy the first two rules, but not the ratio rule;

c. dimensions that satisfy the ratio rule, but not the first two rules.

29. If the ratio of the width to the height is 1.3, what is the ratio of the height to the width?

30. Find the equation of the lines through the origin in your graph. Explain how they are examples of direct variation.

PRACTICE

REVIEW

DIRECT VARIATION

1. Without graphing, tell which of the following lines pass through the origin. Explain. The line containing points

  a. (2, 6) and (4, 12);

  b. (3, 8) and (4, 9);

  c. (6, 5) and (18, 15).

2. A line contains the points (0, 1) and (2, 4). Does it also contain the point (4, 8)? Explain.

3. A line contains the points (2, 5) and (4, 10). Does it also contain the point (200, 500)? Explain.

CREATE AN EQUATION

4. Create an equation that has $x = 3$ as a solution.

5. Create an equation that has $y = -3$ as a solution.

6. Create an equation where x appears on both sides of the equation and

  a. the solution is $x = 0$;

  b. the solution is $x = -1/2$.

CHAPTER 5

The colorful spiral of a lollipop

Coming in this chapter:

Exploration Build as many rectangles as you can with one x^2, ten x-blocks, and any number of yellow blocks.

Build as many rectangles as you can with x^2, 18, and any number of x-blocks.