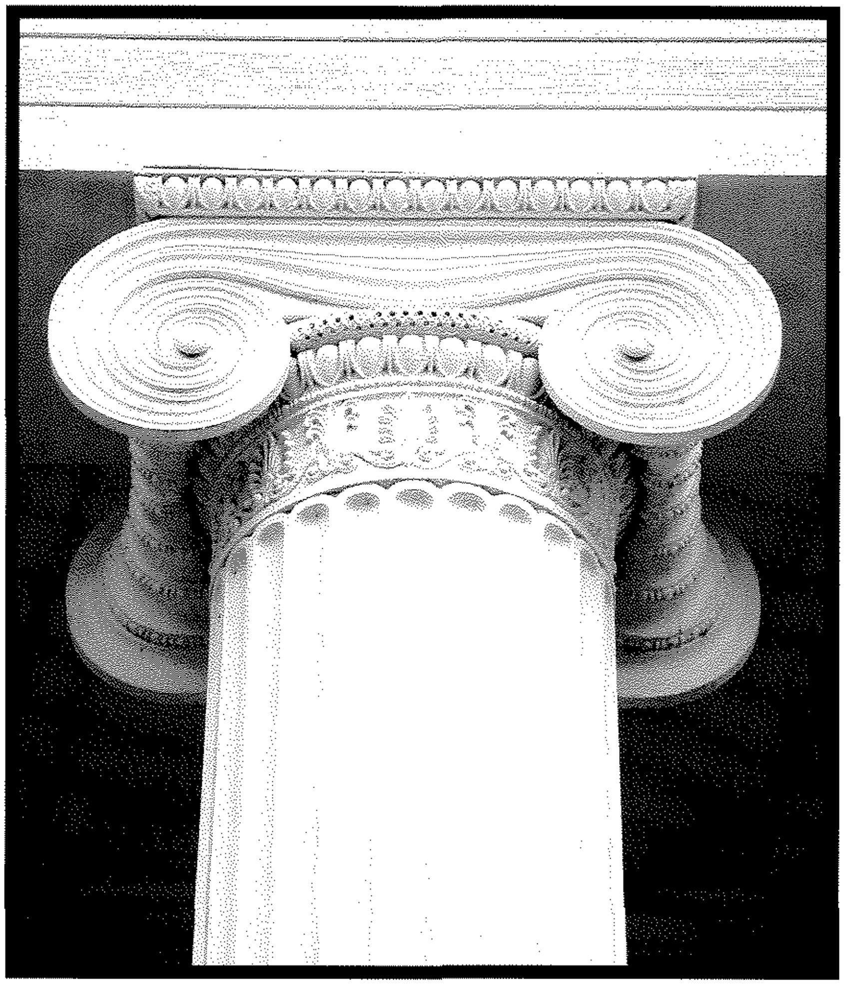
CHAPTER 9



The scroll-like spirals in the capital of a Greek column

Coming in this chapter:

Exploration There are four geoboard segments that start at the origin and have length 5. Find their endpoints. Use this to help you solve the following problem: If you know that two sides of a geoboard triangle are of length 5, what are the possible lengths for the third side?

MEASUREMENT AND SQUARE ROOTS

LESSON 9.1

Distance

You will need:

graph paper

TAXICAB DISTANCE

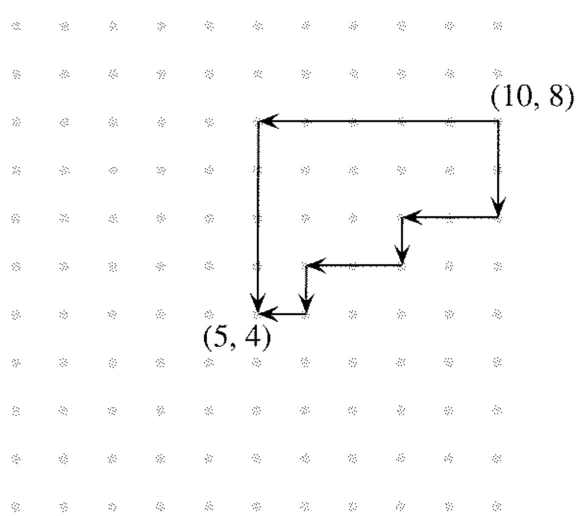
1. Assume you can travel only horizontally and vertically on the Cartesian plane, never letting your x- or y-coordinates decrease.

a. Find at least three ways to get from the origin to (3, 4).

b. Does the travel distance depend on the path you found in part (a) or is it the same for all of them? Explain.

Definition: The *taxicab distance* between two points in the Cartesian plane is the length of the *shortest path* between them that consists of *only* horizontal and vertical segments. Taxicab distance gets its name because it models distance in a city with a network of perpendicular streets.

Example: The taxicab distance from (10, 8) to (5, 4) is 9.



2. What is the taxicab distance between:

a. (1, 2) and (6, 7)?

b. (1, 2) and (1, 7)?

c. (1, 7) and (6, 2)?

d. (-1, -7) and (6, -2)?

e. (1.2, 3.4) and (5.67, 8.9)?

3. a. Find all the points that are at taxicab distance 5 from (5, 5). Sketch them.

b. Describe the shape you found in part (a). Some math teachers call this shape a *taxicab circle*. Explain why.

c. What else might this shape be called?

4. Describe the set of points whose taxicab distance from (5, 5) is

a. greater than 5;

b. less than 5.

TAXICAB vs. EUCLIDEAN DISTANCE

Euclidean distance (named after the ancient Greek mathematician Euclid) is the straight-line distance ("as the crow flies") we studied in a previous lesson.

5. A crow and a taxicab go from the origin to (5, 5). How far does each have to travel?

6. Give examples, if possible, and explain.

a. When are Euclidean and taxicab distances between two points equal?

b. When is Euclidean distance greater than taxicab distance?

c. When is taxicab distance greater?

7. *A straight line is the shortest path between two points.* Explain how this statement is relevant to problem 6.

8. Sketch all the points that are at the same taxicab distance from both (4, 3) and (6, 7).

9. Sketch all the points that are at the same Euclidean distance from both (4, 3) and (6, 7).

10. Find all points P such that:
- the taxicab distance from P to (4, 3) is greater than the taxicab distance from P to (6, 7), **but**
- the Euclidean distance from P to (6, 7) is greater than the Euclidean distance from P to (4, 3).

Explain, using sketches and calculations.

ABSOLUTE VALUE

11. Find the Euclidean distance between:

a. (1, 2) and (6, 2);

b. (6, 2) and (1, 2);

c. (6.7, 3.45) and (8.9, 3.45).

12. Explain in words how to find the distance between (x_1, y) and (x_2, y) if:

a. $x_1 > x_2$; b. $x_1 < x_2$.

If the y-coordinates of two points are the same, the distance between the two can be found by subtracting the x-coordinates. If the result of the subtraction is negative, use its opposite, since distance is always positive. This is called the *absolute value* of the difference.

13. Find the absolute value of the difference between:

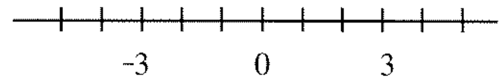
a. 2 and 5; b. 3 and -9;

c. -2 and -5; d. -3 and 9.

14. Explain how you find the distance between two points whose x-coordinates are the same. Give an example.

Definition: The *absolute value* of a number x is the distance from x to 0 on the number line.

Example: The absolute value of 3 is 3. The absolute value of -3 is also 3.



15. Find the absolute value of:

a. 12; b. -1/4.

Notation: The absolute value of a number z is written $|z|$. For example:

$|2| = 2$ $|-2| = 2$

The absolute value of a difference can be written using the same symbol. For example, the absolute value of the difference between a and b is written $|a - b|$.

16. Find the absolute value of:

a. $3 - -5$; b. $-5 - 3$.

17. What is the distance between x and 3 on the number line?

a. Explain in words how to find it.

b. Write a formula, using absolute value notation.

18. Using absolute value notation, the distance between (x_1, y) and (x_2, y) can be written $|x_1 - x_2|$ or $|x_2 - x_1|$. Explain.

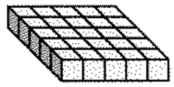
19. Use absolute value notation to write the distance between (x, y_1) and (x, y_2).

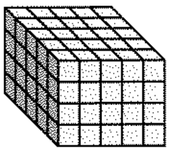
20. Use absolute value notation to write the taxicab distance between (x_1, y_1) and (x_2, y_2).

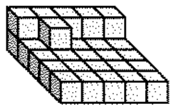
REVIEW SURFACE AREA OF BUILDINGS

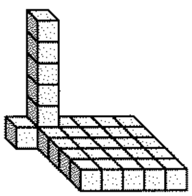
Find the volume and surface area of each of these buildings (including the underside).

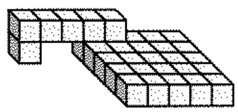
21. 

22. 

23. 

24. 

25. 

26. 

PREVIEW MIXTURES

Tina was thirsty, so Tina and Lana decided to make lemonade. They planned to make a lot, so they could sell some of it at a roadside stand.

Tina started making lemonade using the "taste" method. She added 21 cups of water to 16 cups of lemonade concentrate, but it tasted too lemony. Then she noticed directions on the lemonade package:

> Add water to taste. Most people like a mixture that is 1/5 to 1/4 concentrate.

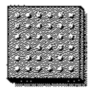
27. How much water should she add to get a mixture that is 1/5 concentrate?

28. Lana tasted the lemonade after Tina had added water to get a mixture that was 1/5 concentrate. It didn't taste lemony enough. How much lemonade concentrate should they add now to get a mixture that is 1/4 concentrate?

LESSON 9.2

The Pythagorean Theorem

You will need:

geoboards 

dot paper 

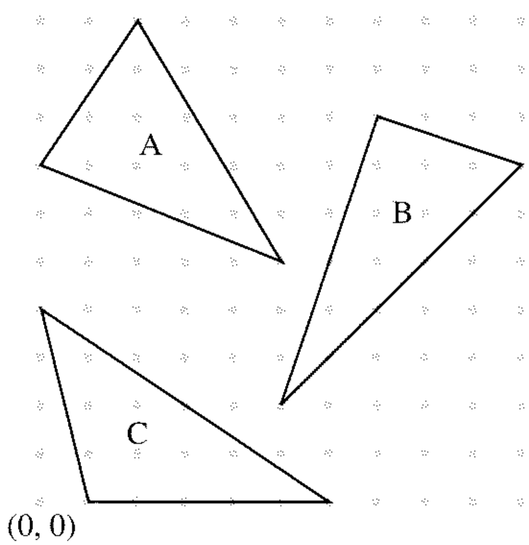
RIGHT TRIANGLES

The corner of a piece of paper can be used to measure a *right angle.*

1. The figure shows three triangles, having a total of nine angles. To do the following problems, you may copy the figure onto your geoboard.

a. Give the coordinates of the vertex of the right angle.

b. Give the coordinates of the vertex of the angle that is greater than a right angle.

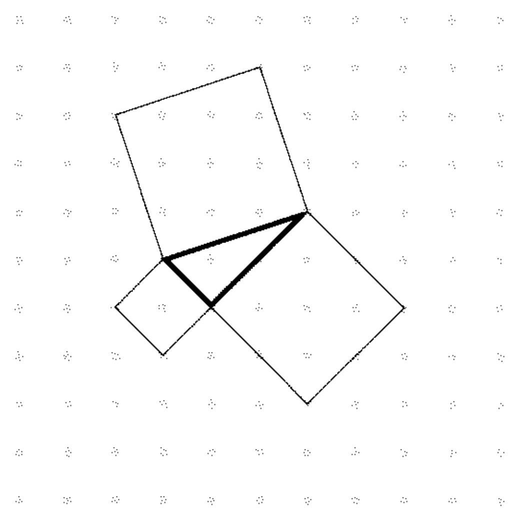


Definitions: An angle that is greater than a right angle is called *obtuse.* An angle that is less than a right angle is called *acute.* A triangle that contains an obtuse angle is called an *obtuse triangle.* A triangle that contains three acute angles is called an *acute triangle.* A triangle that contains a right angle is called a *right triangle.*

2. Which triangle in the preceding figure is acute? Right? Obtuse?

Definition: The two sides forming the right angle in a right triangle are called the *legs.*

The figure shows a right triangle. Three squares have been drawn, one on each of the sides of the triangle.



3. What is the area of each of the three squares?

4. **Exploration** Working with other students, make eight figures like the one above (on geoboards or dot paper). Each figure must be based on a different right triangle. For each one, find the areas of the three squares. Fill out a table like the one below. Study the table for any pattern.

Areas of:

Square on short leg	Square on long leg	Square on hypotenuse
...	...	...

The pattern you probably discovered is called the Pythagorean theorem, after the ancient Greek mathematician Pythagoras. The pattern is about the relationship of the squares on the legs to the square on the hypotenuse for a right triangle.

5. Make an acute triangle on a geoboard or dot paper. Draw a square on each of its sides. Is the sum of the areas of the two smaller squares equal to, greater than, or less than the area of the large square?

6. Repeat problem 5 with an obtuse triangle.

7. Summary Explain the following equation.

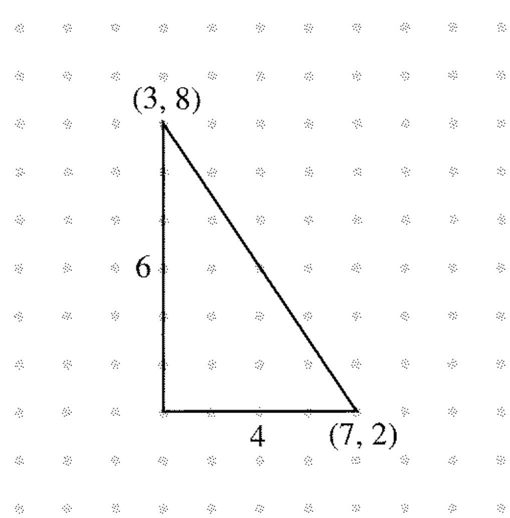
$$\text{leg}^2 + \text{leg}^2 = \text{hyp}^2$$

(Is it true of any triangle? For those triangles for which it is true, what does it mean?)

FINDING DISTANCES FROM COORDINATES

The Pythagorean theorem provides us with a way to find distances in the Cartesian plane. (*Distance* usually refers to Euclidean distance.)

Example: What is the distance between (3, 8) and (7, 2)? First sketch the points.



You can see on the sketch that the length of the legs is 4 for the horizontal leg, and 6 for the vertical leg. Since the triangle shown is a right triangle, we can use the Pythagorean theorem, ($\text{hyp}^2 = \text{leg}^2 + \text{leg}^2$). In this case,

$$\text{distance}^2 = 4^2 + 6^2 = 16 + 36 = 52$$

$$\text{so, distance} = \sqrt{52} = 7.21\ldots$$

8. For the example, if you did not sketch the figure, how could you find the lengths of the legs directly from the coordinates?

Note that the lengths of the legs have been called the *rise* and the *run* when discussing slope. However keep in mind that rise, run, and slope can be positive, negative, or zero, while distances cannot be negative.

9. Consider the two points (3, -4) and (4, -9). Use a sketch if you need to.

a. Find the rise between them.

b. Find the run between them.

c. Find the slope of the line that joins them.

d. Find the taxicab distance between them.

e. Find the Euclidean distance between them.

10. Use any method to find the (Euclidean) distance between:

a. (-1, 2) and (-1, -7);

b. (-1, 2) and (5, 2);

c. (-1, 2) and (5, -7);

d. (-1, 2) and (-1, 2).

11. For which part of problem 10 is the Pythagorean theorem helpful? Explain.

12. Find the distances between:

a. (8, 0) and (0, -8);

b. (-8, 0) and (3, -8);

c. (1.2, 3.4) and (-5.6, 7.89).

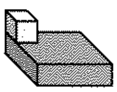
AN OLD PROBLEM

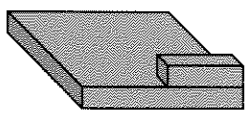
The mathematician Leonardo of Pisa, also known as Fibonacci, posed this problem in 1202.

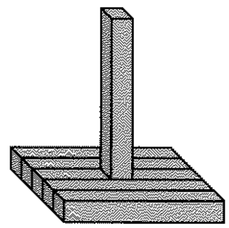
13. Two towers of height 30 paces and 40 paces are 50 paces apart. Between them, at ground level, is a fountain towards which two birds fly from the tops of the towers. They fly at the same rate, and they leave and arrive at the same time. What are the horizontal distances from the fountain to each tower?

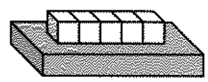
REVIEW *MORE SURFACE AREA*

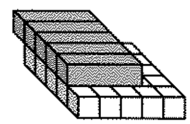
Imagine these buildings are made by gluing Lab Gear blocks together. The surface area is the total area of all the exposed faces, even the bottom of the building. Find the surface areas.

14. 

15. 

16. 

17. 

18. 

LESSON 9.3

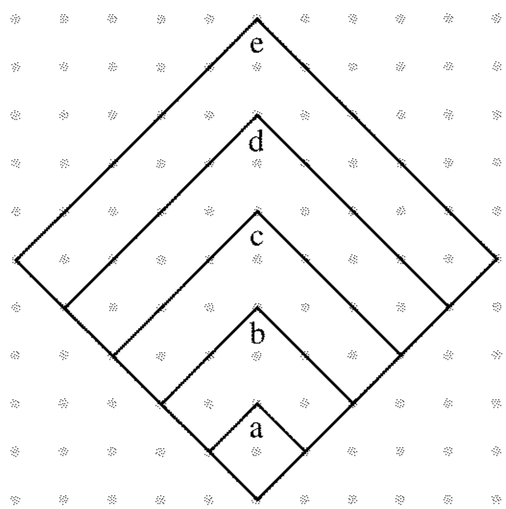
Radicals

You will need:

geoboards

dot paper

SQUARES AND ROOTS



The figure shows five squares. For each one, find

1. its area;
2. its side, written twice: as the square root of the area, and as a decimal number.

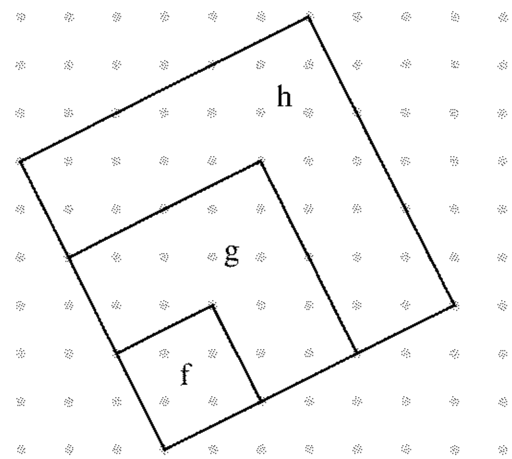
The sides of the larger squares are multiples of the side of the smallest square. For example, square (b) has a side that is equal to two times the side of square (a). You can write,

$$\sqrt{8} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$$

Note that $2\sqrt{2}$ means 2 times $\sqrt{2}$, just as $2x$ means 2 times x. You can check the equation with a calculator.

$$\sqrt{8} = 2.828427125\ldots$$
$$2\sqrt{2} = 2.828427125\ldots$$

3. Write equations about the sides of squares (c), (d), and (e). Check their correctness with a calculator.



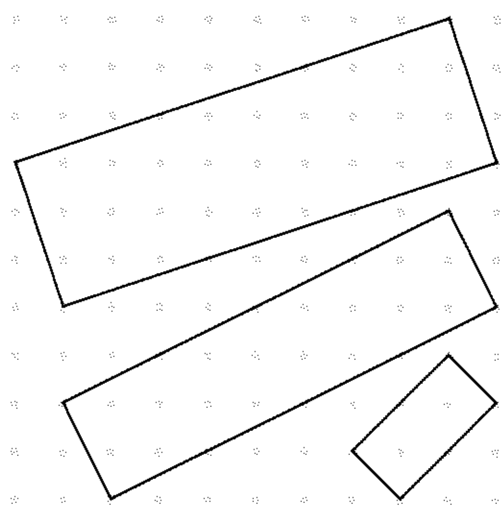
The figure shows three squares. For each one, find

4. its area;
5. its side, written twice: as the square root of the area, and as a decimal number.
6. Write equations involving square roots based on the figure. Check your equations on a calculator.
7. True or False? Use a sketch on dot paper to explain your answers.
   a. $\sqrt{2} + \sqrt{2} = \sqrt{4}$
   b. $4\sqrt{2} = \sqrt{8}$
8. Is $\sqrt{2 + 2} = \sqrt{4}$? Explain.

RECTANGLES AND ROOTS

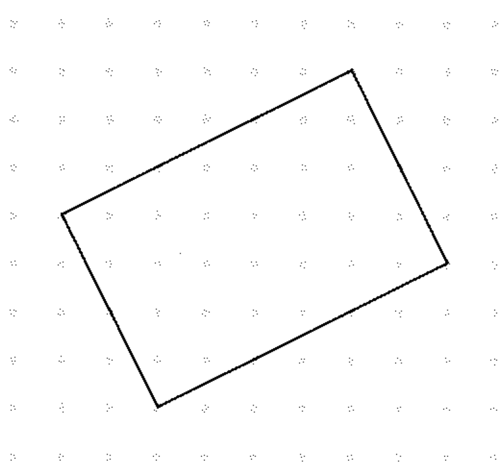
In this section do not use decimal approximations.

9. The figure shows three rectangles. For each one, write *length* · *width* = *area*.



10. For each rectangle above:
   a. What is the side of a square having the same area?
   b. Sketch this square on dot paper.

Some multiplications involving square roots can be modeled by geoboard rectangles. For example, $2\sqrt{5} \cdot 3\sqrt{5}$ is shown in this figure.



11. Find the product of $2\sqrt{5} \cdot 3\sqrt{5}$ by finding the area of the rectangle.

12. Multiply.
   a. $2\sqrt{2} \cdot 3\sqrt{2}$ b. $3\sqrt{2} \cdot 4\sqrt{2}$
   c. $4\sqrt{2} \cdot 5\sqrt{2}$ d. $\sqrt{2} \cdot 2\sqrt{2}$

13. Multiply.
   a. $\sqrt{2} \cdot \sqrt{18}$ b. $\sqrt{18} \cdot \sqrt{50}$
   c. $\sqrt{50} \cdot \sqrt{8}$ d. $\sqrt{8} \cdot \sqrt{32}$

Using the fact that $\sqrt{a} \cdot \sqrt{a} = a$ makes it easy to multiply some quantities involving radicals. For example:

$$6\sqrt{5} \cdot 2\sqrt{5} = 6 \cdot 2 \cdot \sqrt{5} \cdot \sqrt{5} = 12 \cdot 5 = 60$$

14. Multiply.
   a. $5\sqrt{2} \cdot \sqrt{2}$ b. $5\sqrt{2} \cdot 4\sqrt{2}$
   c. $3\sqrt{5} \cdot \sqrt{5}$

15. Explain your answers by using a sketch of a geoboard rectangle.
   a. Is $\sqrt{4} \cdot \sqrt{2} = \sqrt{8}$?
   b. Is $\sqrt{5} \cdot \sqrt{20} = \sqrt{100}$?

MULTIPLYING SQUARE ROOTS

Is it always true that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$? We cannot answer this question in general by making geoboard rectangles. A multiplication like $\sqrt{2} \cdot \sqrt{5}$ cannot be shown that way because it is not possible to find those lengths on the geoboard at a right angle to each other.

16. Guess how to write $\sqrt{2} \cdot \sqrt{5}$ as a square root. Check your guess with a calculator.

17. **Generalization** If a and b are positive,
   a. give a rule for multiplying $\sqrt{a} \cdot \sqrt{b}$;
   b. explain how to multiply $c\sqrt{a} \cdot d\sqrt{b}$.

18. Multiply.
   a. $3\sqrt{5} \cdot 2\sqrt{6}$
   b. $(2\sqrt{11})(-11\sqrt{2})$

SIMPLE RADICAL FORM

Definitions: The square root symbol ($\sqrt{\ }$) is called a *radical sign,* or simply *radical.* A *radical expression* is an expression that includes a radical.

Examples:

$$\sqrt{3},\ 4\sqrt{7},\ 1 + \sqrt{6},\ \text{or}\ \frac{\sqrt{2}}{x}$$

19. Write each of these in at least two ways as the product of two radical expressions.
   a. $\sqrt{70}$ b. $\sqrt{63}$
   c. $6\sqrt{80}$ d. $24\sqrt{105}$

20. Write each of these as the product of two radicals, one of which is the square root of a perfect square.

a. $\sqrt{75}$ b. $\sqrt{45}$

c. $\sqrt{98}$ d. $\sqrt{28}$

Definition: Writing the square root of a whole number as a product of a whole number and the square root of a smallest possible whole number is called putting it in *simple radical form.*

For example, in simple radical form,

$\sqrt{50}$ is $5\sqrt{2}$ $\sqrt{20}$ is $2\sqrt{5}$.

(Note that when using a calculator to find an approximate value, simple radical form is not simpler!)

21. Write in simple radical form.

a. $\sqrt{75}$ b. $\sqrt{45}$

c. $\sqrt{98}$ d. $\sqrt{28}$

GEOBOARD LENGTHS

Since 50 is a little more than 49, $\sqrt{50}$ is a little more than 7. A calculator confirms this: $\sqrt{50} = 7.07\ldots$

22. Estimate the following numbers, and check your answer on a calculator.

a. $\sqrt{65}$ b. $\sqrt{85}$

These numbers may help you with the next problem.

23. Exploration There are 19 geoboard line segments that start at the origin and have length 5, 10, $\sqrt{50}$, $\sqrt{65}$, or $\sqrt{85}$. Find them, and mark their endpoints on dot paper.

24. If you know two sides of a geoboard triangle are of length 5, what are the possibilities for length for the third side?

25. Repeat problem 24 for the following side lengths.

a. 10 b. $\sqrt{50}$

c. $\sqrt{65}$ d. $\sqrt{85}$

LESSON 9.4

Radical Operations

You will need:

- geoboards
- dot paper

MULTIPLICATION

1. **Exploration** Using only multiplication, write at least three radical expressions that equal each of the following.

a. $2\sqrt{3}$ b. 6

Even though you are often asked to simplify expressions, it is sometimes just as important to know how to "complicate" them. For example, $3\sqrt{7}$ is equivalent to all these radical expressions.

$\sqrt{9}\sqrt{7}$ $\sqrt{9 \cdot 7}$ $\sqrt{63}$

$\sqrt{3}\sqrt{3}\sqrt{7}$ $\sqrt{3}\,\sqrt{21}$

2. Write at least two other radical expressions equivalent to:

a. $5\sqrt{2}$; b. $2\sqrt{5}$;

c. $6\sqrt{10}$; d. $10\sqrt{6}$.

3. Write each as the square root of a number. (For example, $3\sqrt{7} = \sqrt{63}$.)

a. $2\sqrt{2}$ b. $2\sqrt{7}$

c. $5\sqrt{6}$ d. $4\sqrt{3}$

4. Write each as the product of as many square roots as possible. (For example, $3\sqrt{6} = \sqrt{3} \cdot \sqrt{3} \cdot \sqrt{2} \cdot \sqrt{3}$.)

a. $5\sqrt{10}$ b. $7\sqrt{5}$

c. $\sqrt{30}$ d. $10\sqrt{22}$

5. What number times $\sqrt{6}$ equals $3\sqrt{10}$?

To answer problem 5, Tina wrote:

$\_\_\_\_\_ \cdot \sqrt{6} = 3\sqrt{10}$

$\_\_\_\_\_ \cdot \sqrt{3}\sqrt{2} = \sqrt{3}\sqrt{3}\sqrt{2}\sqrt{5}$

"First I wrote everything as a product of square roots," she explained. "Then it was easy to see that the missing factors were $\sqrt{5}$ and $\sqrt{3}$, so the answer must be $\sqrt{15}$."

Erin politely told Tina that her method seemed unnecessarily complicated. Erin wrote:

$\_\_\_\_\_ \cdot \sqrt{6} = 3\sqrt{10}$

$\_\_\_\_\_ \cdot \sqrt{6} = \sqrt{9}\sqrt{10}$

$\_\_\_\_\_ \cdot \sqrt{6} = \sqrt{90}$

"My goal was to write $3\sqrt{10}$ as the square root of something. Once I found that $3\sqrt{10} = \sqrt{90}$, it was easy from there. I could use the rule that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ to see that the answer was $\sqrt{15}$," she explained.

6. What number times $2\sqrt{10}$ equals $10\sqrt{2}$? Find the answer by using

a. Tina's method;

b. Erin's method.

7. What number times $\sqrt{8}$ equals 4?

8. What number times $2\sqrt{2}$ equals $4\sqrt{3}$?

DIVISION

9. Divide 5 by $2\sqrt{5}$.

"That's not fair," said Tina. "Ms. Kem never taught us to divide with radicals." "That's true," said Erin, "but we know that multiplication and division are inverse operations." She wrote:

$\_\_\_\_\_ \cdot 2\sqrt{5} = 5$

$\_\_\_\_\_ \cdot \sqrt{4}\sqrt{5} = \sqrt{25}$

10. Finish solving the problem using Erin's method.

Another way to solve this problem is to use the following trick: *Write an equivalent fraction without a square root in the denominator.* In this case, we multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{5}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{5 \cdot \sqrt{5}}{2 \cdot 5} = \frac{\sqrt{5}}{2}$$

11. Explain why $\sqrt{5}$ was chosen as the number by which to multiply.

12. Divide.

a. $\frac{3}{2\sqrt{6}}$ b. $\frac{24}{\sqrt{6}}$ c. $\frac{3\sqrt{10}}{5\sqrt{3}}$ d. $\frac{5\sqrt{3}}{3\sqrt{10}}$

MORE ON SIMPLE RADICAL FORM

Using the fact that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, we can write $\sqrt{63}$ as $\sqrt{21} \cdot \sqrt{3}$. We can also write it as $\sqrt{9} \cdot \sqrt{7}$, which is especially convenient because 9 is a perfect square. Therefore:

$$\sqrt{63} = \sqrt{9} \cdot \sqrt{7} = 3\sqrt{7}.$$

This last expression is in simplest radical form.

13. Write in simple radical form.

a. $\sqrt{200}$ b. $\sqrt{147}$

c. $\sqrt{700}$ d. $\sqrt{275}$

ADDITION AND SUBTRACTION

14. Use dot paper to illustrate the addition $\sqrt{5} + 2\sqrt{5}$.

15. Using the figure you made in problem 14, explain how to decide which of the two equations $\sqrt{5} + \sqrt{20} = \sqrt{25}$ and $\sqrt{5} + \sqrt{20} = \sqrt{45}$ is correct.

16. Check your answer to problem 15 with a calculator.

17. True or False? Explain.

a. $16 + 9 = 25$

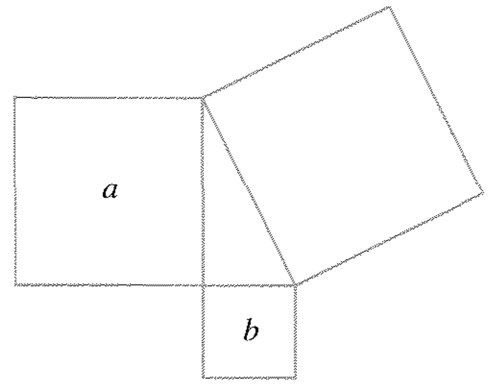
b. $\sqrt{16 + 9} = \sqrt{25}$

c. $\sqrt{16} + \sqrt{9} = \sqrt{25}$

d. $\sqrt{16} + \sqrt{9} = \sqrt{16 + 9}$

18. If a and b are positive numbers, is it always, sometimes, or never true that $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$? Explain, with examples.

The figure shows a right triangle, a square having area a, a square having area b, and a third square.



19. In terms of a and b,

a. what is the area of the third square? Explain.

b. What are the sides of the triangle?

20. If a and b are positive numbers, is it always, sometimes, or never true that $\sqrt{a} + \sqrt{b} > \sqrt{a + b}$? Explain, using the figure.

As you see, sums of radical expressions cannot usually be simplified. However, in some cases, simple radical form can help.

21. Simplify, then add or subtract.

a. $\sqrt{18} + \sqrt{32}$

b. $\sqrt{18} - 4\sqrt{20}$

c. $\sqrt{60} - \sqrt{135}$

d. $\sqrt{45} + \sqrt{125}$

You can add or subtract square roots only if they are the roots of the same number. This is similar to combining like terms when adding polynomials.

22. Simplify, then add or subtract, if possible.

a. $5 + 5\sqrt{68} + \sqrt{17}$

b. $6 - 6\sqrt{15} + \sqrt{90}$

c. $\sqrt{8} + \sqrt{16} + \sqrt{32} - \sqrt{64}$

d. $\sqrt{10} + \sqrt{20} - \sqrt{30} + \sqrt{40} - \sqrt{50}$

THINKING WRITING

9.A Geoboard Distances

You will need:

geoboards

dot paper

1. Find the distance between the origin and each geoboard peg. Use radical expressions for your answers, not decimal approximations. Arrange your results in a table like the one below, with the peg coordinates along the sides. In each space write the peg's distance from the origin. Some examples have been entered to get you started. To speed this up, work with a partner and look for patterns.

10											
9											
8											
7											
6											
5	5										
4				5							
3					5						
2		$\sqrt{5}$									
1			$\sqrt{5}$								
0						5					
	0	1	2	3	4	5	6	7	8	9	10

2. Describe the patterns you see in the table.

3. Find the numbers in the table that are not in simple radical form. Put them in that form and describe the patterns you notice.

4. What is the distance from the origin to the furthest peg on a geoboard having dimensions

a. 20 by 20? b. n by n?

c. 20 by 30? d. m by n?

5. On a 20-by-20 geoboard, what would be the largest multiple of:

a. $\sqrt{2}$? b. $\sqrt{5}$?

c. $\sqrt{10}$?

6. Notice that all the multiples of $\sqrt{2}$ lie on a line. What is the slope of this line?

7. a. Why are there two lines containing multiples of $\sqrt{5}$?

b. What are the slopes of these lines?

8. Repeat problem 7(b) for multiples of:

a. $\sqrt{10}$; b. $\sqrt{17}$.

9. List the geoboard distances that are on the line through the origin having slope

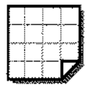
a. 5; b. 3/4.

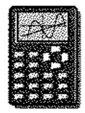
10. **Report** Summarize your results from this lesson. Describe and explain the patterns you noticed and the generalizations you made.

LESSON 9.5

The Square Root Function

You will need:

graph paper 

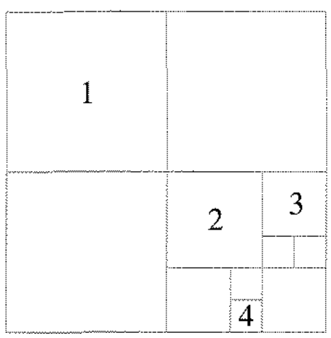
graphing calculator (optional) 

ROOTS OF NUMBERS < 1

The large square in the figure has dimensions 1-by-1 unit. It is divided into 11 smaller squares. For the square on the top left, you could write the following equations, relating the length of its side to its area.

a. $1/4 = (1/2)^2$ b. $1/2 = \sqrt{1/4}$

c. $0.25 = 0.5^2$ d. $0.5^2 = \sqrt{0.25}$



1. Explain the above equations.

2-4. For each numbered smaller square, write equations of the form:

a. area = side^2, using fractions

b. side = $\sqrt{\text{area}}$, using fractions

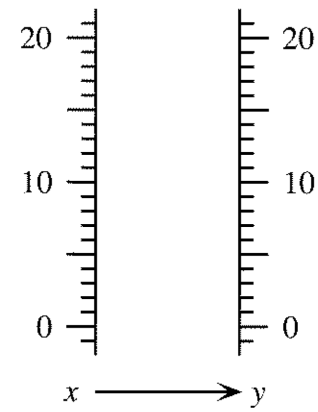
c. area = side^2, using decimals

d. side = $\sqrt{\text{area}}$, using decimals

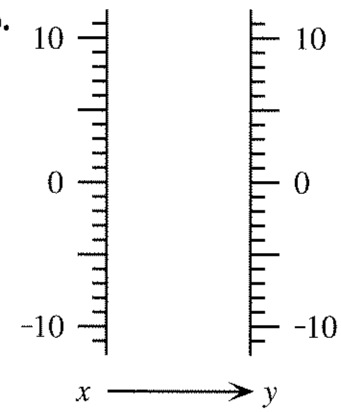
DIAGRAMS FOR SQUARES AND ROOTS

The function diagrams for the same function could look quite different with different scales.

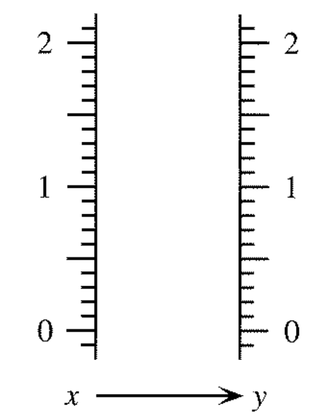
a.



b.



c.

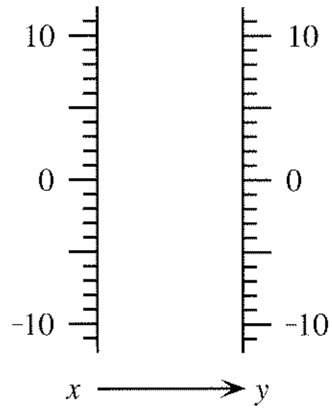


5. Make three function diagrams for the function $y = x^2$, using the scales given in the figure. Use nine in-out pairs for each.

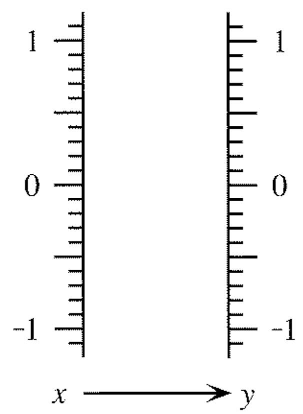
Problems 6 and 7 are about $y = x^2$.

6. In the function diagrams below, how far would you have to extend the y-number line in the positive direction so that every value you can see on the x-number line has a corresponding y-value on the diagram? How about in the negative direction?

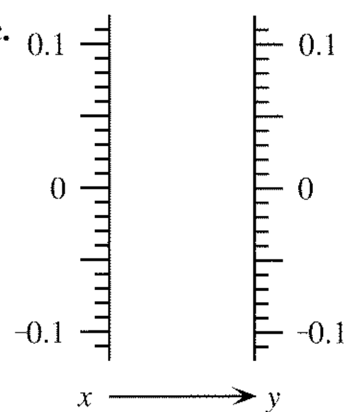
a.



b.

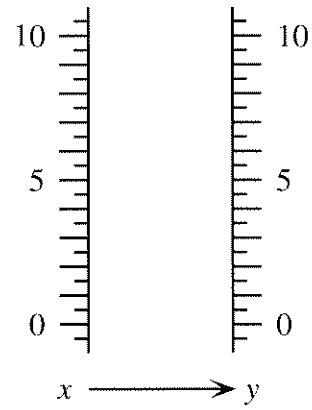


c.

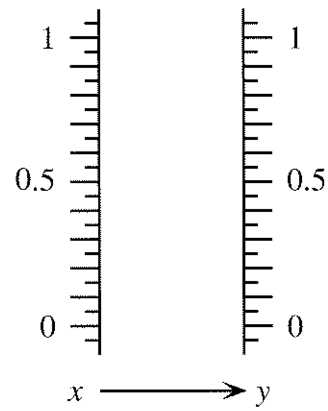


7. In the function diagrams below, how far would you have to extend the x-number line, if at all, so that every value you can see on the y-number line has a corresponding x-value on the diagram?

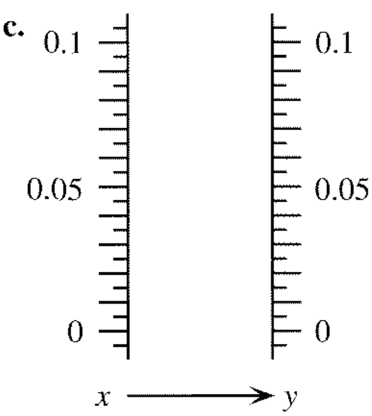
a.



b.



c.



Definitions: The *domain* of a function is the set of the values that the input can take. The *range* of a function is the set of the values the output can take.

Example: The domain of $y = x^2$ is *all numbers*, since any number can be squared.

8. Explain why the range of the function $y = x^2$ is all nonnegative numbers.

Notation: $\sqrt{x}$ represents the nonnegative number whose square is x.

Example: $\sqrt{4}$ represents only 2, even though $(-2)^2$ also equals 4. However we can write

$$-2 = -\sqrt{4}.$$

9. Using the same scales as in problem 5, make three function diagrams for the function $y = \sqrt{x}$.

10. For which scale is the function diagram not a mirror image of the corresponding one for $y = x^2$? Explain.

11. What are the domain and the range of the square root function? Explain.

12. To *be* or to *have*, that is the question.
a. Which numbers have a square root?
b. Which numbers have a square?
c. Which numbers can be a square?
d. Which numbers can be a square root?

GRAPHS FOR SQUARES AND ROOTS

13. Make tables of at least eight (x, y) pairs each for these two functions and graph them on the same axes. Use three values of x between 0 and 1, as well as negative values and whole numbers.
a. $y = x^2$ b. $y = \sqrt{x}$

14. On the same axes, graph the line $y = x$.

15. The curve representing $y = x^2$ is called a *parabola.* What would you call the curve representing $y = \sqrt{x}$?

16. Which of your three graphs grows
a. faster and faster?
b. more and more slowly?
c. always at the same rate?

17. If extended to the right, how high would the curve representing $y = \sqrt{x}$ go? (Can you find an x such that $\sqrt{x}$ is larger than 100? Than 1000?) Explain.

18.
a. What numbers are greater than their squares?
b. What numbers are less than their square roots?
c. What numbers are equal to their square roots?
d. What numbers are equal to their squares?

19. Solve the equations.
a. $x^2 = 5$ b. $x^2 = -5$
c. $\sqrt{x} = 5$ d. $\sqrt{x} = -5$
e. $-\sqrt{x} = -5$

20. Solve the inequalities. (Be careful! Some have compound solutions.)
a. $x^2 < 4$ b. $\sqrt{x} < 2$
c. $x^2 < \sqrt{x}$ d. $x^2 > 6$

21. Solve the equations and inequalities.
a. $P^2 = 456$ b. $P^2 < 456$
c. $\sqrt{K} = 789$ d. $\sqrt{K} < 789$

22. Report Summarize what you know about the functions $y = x$, $y = x^2$, and $y = \sqrt{x}$. Use graphs, diagrams, and examples. Include answers to these questions.
- Which is greatest and which is least among x, x^2, or $\sqrt{x}$? Explain how the answer depends on the value of x.
- What are the domains and ranges of these three functions?

23. Sketch the graphs of $y = \sqrt{x}$ and $y = \sqrt{-x}$. Think about domain and range!

MORE SQUARE ROOT GRAPHS

Use a graphing calculator if you have one.

24. Graph these equations on the same pair of axes.
    a. $y = 4\sqrt{x}$
    b. $y = \sqrt{4x}$
    c. $y = \sqrt{4}\sqrt{x}$

25. In problem 24, which graphs are the same? Explain.

26. Graph these equations on the same pair of axes.
    a. $y = \sqrt{x + 9}$
    b. $y = \sqrt{x} + 3$
    c. $y = \sqrt{x} + \sqrt{9}$

27. In problem 26, which graphs are the same? Explain.

PUZZLES *PACKING SQUARES*

28. A 10-by-10 square can be divided into 11 smaller ones, with no overlaps and no space left over (as in the figure at the very beginning of the lesson). Divide each of the following squares into 11 smaller squares. (The side lengths of the smaller squares must be integers.)
    a. 11-by-11
    b. 12-by-12
    c. 13-by-13

DISCOVERY *WALKING DISTANCE*

Use graph paper as the map of a city. The horizontal and vertical lines represent streets.

29. In your group, agree on the location of various buildings, such as a supermarket, a hospital, a school, a fast food outlet, a bank, etc. Mark them on dot paper. Make a list of their coordinates.

    Make up a problem about finding a good place for a couple to live in your city. Assume that they do not want to drive, and that they work in different places. Each student should choose a different job for each member of the couple.
    a. Where should they live if they want to minimize the total amount of distance walked to work?
    b. Where should they live if, in addition, they want to walk equal amounts?

LESSON 9.6

Midpoints

You will need:

graph paper

MEETING HALFWAY

1. Linda works at the corner of Galbrae Avenue and 15th Street. Micaelia works at the corner of Galbrae Avenue and 38th Street. The streets between 15th and 38th are all consecutively numbered streets. Linda and Micaelia agree to meet after work. If they both want to walk the same distance, where should they meet?

2. Change Micaelia's workplace in problem 1. Make her meeting place with Linda at a street corner, not the middle of a block.

3. For what values of n is the halfway point between 15th Street and nth Street in the middle of the block, and for what values is it at a street corner?

4. Find the point on the number line halfway between:

   a. 1.5 and 6.8; b. 1/3 and 1/2.

5. Describe how to find the point on the number line halfway between a and b. Use a sketch and explain.

6. Explain how to find the point on the number line

   a. 1/3 of the way from 4 to 6;

   b. 1/4 of the way from 4 to 7.

FINDING A FORMULA

Sue and Ruth were trying to find the number halfway between 5 and 11.4. Ruth used this method: First she found the distance between 11.4 and 5, which is 6.4. Next she took half of that, which is 3.2. Last she added 3.2 to 5.

7. Use a sketch of the number line to explain Ruth's method.

8. If $B > A$, what is the distance between A and B on the number line? What is half that distance?

9. The formula for Ruth's method is

$$\text{midpoint} = \frac{B - A}{2} + A.$$

   Explain.

10. Ruth's formula can be rewritten as two fractions with a common denominator.

$$\text{midpoint} = \frac{B - A}{2} + \frac{2A}{2}$$

   Write it as one fraction in lowest terms.

11. Explain the formula you found in problem 10 in words.

12. Sue's method for finding the midpoint between two points on the number line is to take the average of the two points. Does that method work? Test it on some examples, and explain what you find out.

13. **Summary** Compare Ruth's method with Sue's method. Use examples, sketches, and algebra. Does either method work all the time? Which one do you prefer? Do they work when A and/or B are negative?

THE MIDPOINT OF LINEAR GROWTH

Between ages 10 and 12, Sue's growth in height was approximately linear as a function of age. This means that the rate of change of height per year was approximately constant.

Sue's Growth (Height)

Age (years)	Height (cm)
10	146
11	—
12	161

14. Estimate Sue's height at age 11.

15. Based on the data, do you think her weight increased linearly as a function of age? If so, estimate her weight at ages 10½ and 11½.

Sue's Growth (Weight)

Age (years)	Weight (lbs)
10	90
11	101
12	112

16. Joel kept a record of his height and weight. When he was 5′5″ tall, he weighed 130 pounds. When he was 5′7″ tall, he weighed 142 pounds. If his weight increased as a linear function of his height, how much did he weigh when he was 5′6″ tall?

MIDPOINT OF A LINE SEGMENT

17. On a graph, plot and label the midpoint of the segment joining each pair of points.

a. (5, 3) and (8, 7)

b. (-5, -3) and (8, -7)

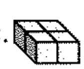
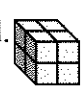
c. (-5.5, 3.5) and (8, 7)

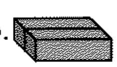
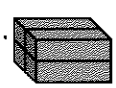
d. (1/4, 3) and (3/4, -7)

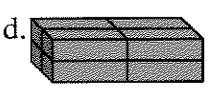
18. Using a sketch, explain how to find the coordinates of the midpoint of the segment joining the points (a, b) and (c, d). Check your method for positive and negative numbers. Try to write a formula.

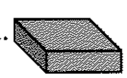
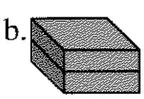
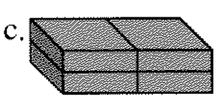
PREVIEW *SURFACE AREA SEQUENCES*

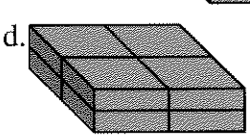
For each sequence of buildings, find the volume and surface area of the first four buildings. Then, describe and sketch the fifth building, and find its volume and surface area.

19. a.  b.  c.  d. 

20. a.  b.  c. 

d. 

21. a.  b.  c. 

d. 

LESSON 9.7

Halfway Measures

TWO ACCOUNTS

1. **Exploration** Janet and Marne had savings accounts. Marne was earning simple interest, and Janet was earning compound interest. Surprisingly, both accounts grew from \$650.00 to \$805.24 in four years. What was the annual interest rate for each account?

2. How much money was in each account after two years?

3. One account increased by the same *amount* every two years. What was the amount?

4. The other account increased by the same *percent* every two years. What was the percent?

5. **Summary** One account was an example of linear growth, the other was an example of exponential growth. In equal time intervals, one account showed constant differences, while the other showed constant ratios. Explain.

THE MIDPOINT OF EXPONENTIAL GROWTH

Dick and Stan had data about the population of their school. There were 325 students in 1980 and 742 students in 1988. They wanted to estimate the population in 1984.

Dick assumed that the population had grown linearly. This means that for equal time intervals the difference in population would be the same. Algebraically,

$$P_{1984} - P_{1980} = P_{1988} - P_{1984}.$$

6. Use algebra to find P_{1984}.

7. If Dick's assumption was correct, what was the population in 1986?

Stan assumed that the population had grown exponentially. This means that for equal time intervals, the population ratios would be the same. Algebraically,

$$\frac{P_{1984}}{P_{1980}} = \frac{P_{1988}}{P_{1984}}$$

$$\frac{P_{1984}}{325} = \frac{742}{P_{1984}}$$

8. Solve for P_{1984}. (Hint: Multiply both sides by 325 and then by P_{1984}.)

9. If Stan's assumption was correct, what was the population in 1986? Explain your reasoning and show your calculations.

10. Assume Stan's assumption was correct and also that the population grew at the same rate from 1980 to 1992. Make a table showing an estimate of the population at two-year intervals during this time period.

LINEAR OR EXPONENTIAL?

Solve these problems in two ways, assuming

a. that the growth is linear;

b. that the growth is exponential.

c. Discuss which assumption is more reasonable, or whether neither one is credible.

11. A tree was 6 feet high in 1930 and 21 feet high in 1980. How high was it in 1955?

12. A tumor was estimated to weigh about 4 grams in January and 7 grams six months later. If it continued to grow in the same way, how much would it weigh after three more months?

13. **Generalization** A growing population is P_1 at a certain time and P_2 at a later time. Use algebra to find its size halfway between these two times, assuming
   a. linear growth;
   b. exponential growth.

USING AN EQUATION

A population grew from 1000 to 2197 in three years.

14. Assume linear growth.
   a. How much did the population grow each year?
   b. Make a table showing the population at the end of one, two, three, and four years.
   c. Write an equation expressing the population as a function of the number of years.
   d. Use the equation to find out the population after 27 months. (Hint: First figure out how many years that is.)

15. Assume exponential growth.
   a. By how much was the population multiplied each year?
   b. Make a table showing the population at the end of one, two, three, and four years.
   c. Write an equation relating the population to the number of years.

Your equation should be in the form $P = 1000b^x$, with x indicating the number of years.

16. Use the equation and your calculator to find the population after:
   a. 27 months; b. 2.5 years;
   c. 1 month.

REVIEW/PREVIEW

CALCULATOR PREDICTIONS

17. a. Predict how your calculator will respond if you try to use it to compute $\sqrt{-9}$.
   b. Explain your prediction.
   c. Check whether you were right.

For each problem, 18-24, two expressions are given.
   a. Predict which is greater or whether they are equal.
   b. Explain your prediction.
   c. Use your calculator to check whether you were right.

18. $\sqrt{2} + \sqrt{8}$ or $\sqrt{18}$
19. $\sqrt{27}$ or $3\sqrt{3}$
20. $2\sqrt{3}$ or $\sqrt{2 \cdot 3}$
21. $\sqrt{3} + \sqrt{3}$ or $\sqrt{6}$
22. $\sqrt{2}/\sqrt{3}$ or $\sqrt{2/3}$
23. $\sqrt{2}\sqrt{3}$ or $\sqrt{2 \cdot 3}$
24. $\sqrt{3} + \sqrt{3} + \sqrt{3}$ or $3\sqrt{3}$
25.
   a. Predict how your calculator will respond if you try to use it to compute $49^{.5}$ (49 to the power one-half).
   b. Explain your prediction.
   c. Use your calculator to check whether you were right.

LESSON 9.8

The Exponent 1/2

THE HALFWAY GROWTH FACTOR

1. A bacterial population is growing exponentially. It is multiplied by nine every day.

a. Copy and complete the table of the population at half-day intervals.

Time	Population
0	100
0.5	—
1	900
1.5	—
2	—

b. Write an equation giving the population as a function of time (measured in days).

2. Repeat problem 1 for a population that is multiplied by 25 every day.

3. For problems 1 and 2:

a. By how much was the population multiplied in half a day?

b. How are these numbers related to the equation?

4. A tumor that is growing exponentially triples in ten years. By how much is it multiplied in five years?

5. **Generalization** An exponentially growing tumor is multiplied in size by B every ten years and by H every five years. How are B and H related? Explain.

A FRACTIONAL EXPONENT

6. Find x.

a. $2^5 \cdot 2^5 = 2^x$

b. $2^3 \cdot 2^3 = x^6$

c. $(2^4)^2 = 2^x$

7. Find x.

a. $9^x \cdot 9^3 = 9^6$

b. $9^x \cdot 9^x = 9^2$

c. $9^x \cdot 9^x = 9^1$

d. $B^x \cdot B^x = B^1$

8. Find x.

a. $(9^x)^2 = 9^6$ b. $(9^x)^2 = 9^1$

c. $(B^x)^2 = B^6$ d. $(B^x)^2 = B^1$

9. Problems 6-8 suggest a meaning for the exponent 1/2. Explain it.

10. Using this meaning of the exponent 1/2, find the following. (Avoid using a calculator if you can.)

a. $16^{\frac{1}{2}}$ b. $400^{\frac{1}{2}}$

c. $25^{-\frac{1}{2}}$ d. $2^{\frac{1}{2}}$

11. Does it make sense to use the exponent 1/2 in the equations you found in problems 1 and 2? Explain your answer.

12. A colony of bacteria was growing exponentially. It weighed 6 grams at noon and 15 grams at 8 P.M. How much did it weigh at 4 P.M.? Explain.

LAWS OF EXPONENTS AND RADICAL RULES

Rules for operations with radicals can be derived from laws of exponents using the fact that

$$x^{\frac{1}{2}} = \sqrt{x}.$$

The following rules assume a and b are nonnegative.

Exponent Rule	Radical Rule
$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^1$	$\sqrt{a}\,\sqrt{a} = a$
$a^{\frac{1}{2}} \cdot b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$	$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$
$\frac{a^1}{a^{\frac{1}{2}}} = a^{\frac{1}{2}}$	$\frac{a}{\sqrt{a}} = \sqrt{a}$
$\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$	$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

13. Check all the radical rules by using $a = 16$ and $b = 9$.

The last rule is especially useful for simplifying rational expressions involving radicals. To be in simple radical form, an expression cannot have any radicals in the denominator or fractions under the radical sign.

Examples:

$$\frac{\sqrt{16}}{\sqrt{8}} = \sqrt{\frac{16}{8}} = \sqrt{2}$$

$$\sqrt{\frac{144}{169}} = \frac{\sqrt{144}}{\sqrt{169}} = \frac{12}{13}$$

$$\frac{\sqrt{48}}{\sqrt{32}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$$

14. Write problems 14-15 in simple radical form. You can check the answers on your calculator.

a. $\frac{\sqrt{60}}{\sqrt{30}}$ b. $\frac{\sqrt{450}}{\sqrt{18}}$

c. $\frac{\sqrt{18}}{\sqrt{2}}$ d. $\frac{\sqrt{20}}{\sqrt{5}}$

15. a. $\sqrt{\frac{25}{125}}$ b. $\sqrt{\frac{32}{48}}$

c. $\sqrt{\frac{3}{75}}$ d. $\sqrt{\frac{1}{12}}$

SQUARE ROOTS OF POWERS

16. Exploration Use your calculator to make a list of the square roots of the powers of ten, from $\sqrt{10^1}$ to $\sqrt{10^{10}}$. Explain any pattern you discover.

17. Explain the pattern you found in problem 16 by using a law of exponents and the exponent 1/2. (Hint: It is not one of the laws listed before problem 13.)

18. Write in simple radical form.

a. $\sqrt{9(10^8)}$ b. $\sqrt{4(10^7)}$

c. $\sqrt{3(10^6)}$ d. $\sqrt{2(10^5)}$

CHALLENGE ESTIMATING POPULATION

19. The population of California was 3,426,861 in 1920 and 15,717,204 in 1960. Assume it grew exponentially and estimate the population in:

a. 1940; b. 1949.

THINKING WRITING

9.B Skidding Distance

Police use a formula to estimate the speed a car was traveling before an accident by measuring its skid marks. This is the formula.

$$S = \sqrt{30df}$$

S is the speed the car was traveling (in mph).

d is the distance the car skidded (in feet).

f is a special number (called the *coefficient of friction*) that depends on the road surface and road conditions.

The number f is determined by the police when they investigate an accident. For a dry tar road, f is usually about 1.0, so the formula is

$$S = \sqrt{30d(1.0)} \quad \text{(dry tar road).}$$

For a wet tar road, f is about 0.5, so the formula is

$$S = \sqrt{30d(0.5)} \quad \text{(wet tar road).}$$

1. Make tables of values and a graph to show speed as a function of the length of the skid marks. Put both curves on the same axes and use a range for d that will give you values of S up to 125 mph.

2. Why is the coefficient of friction less for a wet road than for a dry road? How does that affect the graph?

Police Report

Weather	Skid marks (ft)
wet	112
dry	321
wet	459
wet	173
dry	100
dry	132

3. This table shows a summary of accidents from a police report. All the accidents took place on tar roads. Use formulas or graphs to estimate how fast the cars were going. Explain how you made your estimates.

4. A police report stated that a car had left 150-foot skid marks on a tar road, but the report did not state the weather. Estimate how fast the car was probably traveling if the road had been wet. Then estimate the speed if the road had been dry.

5. There are two sets of skid marks on the same road. The second set is twice as long as the first. Do you think the second car was going twice as fast as the first? If not, was it going less than twice as fast or more than twice as fast? Explain.

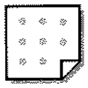
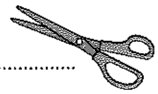
6. The coefficient of friction for a dry concrete road is about 0.8 and for a wet concrete road about 0.4. If a car had been traveling at 50 mph before it skidded, estimate the lengths of skid marks it would have left on each type of road (tar or concrete) and in each type of weather (wet or dry). Compare your answers and comment on the differences you find.

7. Report Imagine that you are responsible for giving a lecture on skidding distance to a class of police cadets who are being prepared to join the highway patrol. You are asked to provide an illustrated two-to-three-page report summarizing the information that you think is important for them to know. Use examples. You may also make a poster to help make your talk more interesting and understandable.

LESSON 9.9

Radical Expressions

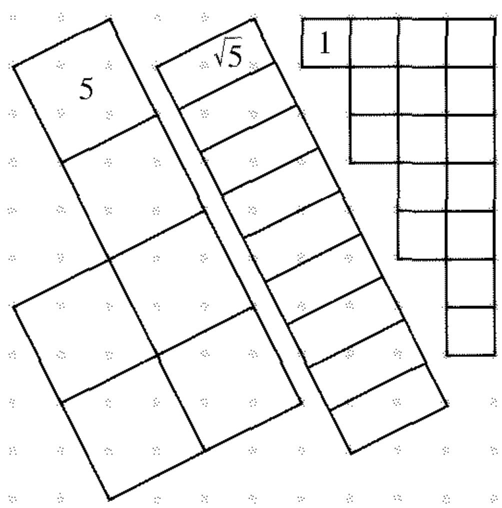
You will need:

- dot paper 
- scissors 

MAKING RADICAL GEAR

This figure shows how to make *radical gear* from dot paper, to help model multiplications like

$$2\sqrt{5} \cdot (\sqrt{5} + 2).$$



Draw some radical gear on dot paper. Cut it out, then use it in the corner piece to do these multiplications.

1. Multiply.
  a. $2\sqrt{5} \cdot (\sqrt{5} + 2)$
  b. $\sqrt{5} \cdot (2\sqrt{5} + 2)$
  c. $4\sqrt{5} \cdot (\sqrt{5} - 1)$
  d. $3\sqrt{5} \cdot (2\sqrt{5} - 1)$

2. Multiply.
  a. $(2\sqrt{5} + 1) \cdot (\sqrt{5} + 2)$
  b. $(2 + \sqrt{5}) \cdot (\sqrt{5} + 2)$
  c. $(2\sqrt{5}) \cdot (2\sqrt{5})$
  d. $(2\sqrt{5})(2 + \sqrt{5})$

3. Multiply.
  a. $(2\sqrt{5} + 1) \cdot (2\sqrt{5} - 1)$
  b. $(\sqrt{5} + 1) \cdot (\sqrt{5} - 1)$
  c. $(3\sqrt{5} - 1) \cdot (\sqrt{5} + 1)$
  d. $3 + \sqrt{5}(2\sqrt{5} - 1)$

APPLYING THE DISTRIBUTIVE LAW

Rule: As you probably noticed, when multiplying radical expressions, *the radicals are handled as if they were variables*.

Example: You can set up a table to multiply $(\sqrt{3} - 2)(\sqrt{2} - \sqrt{3})$.

	$\sqrt{3}$	-2
$\sqrt{2}$	$\sqrt{6}$	$-2\sqrt{2}$
$-\sqrt{3}$	-3	$2\sqrt{3}$

So the product is $\sqrt{6} - 2\sqrt{2} + 2\sqrt{3} - 3$.

4. Multiply.
  a. $7\sqrt{3} \cdot (\sqrt{6} - \sqrt{3})$
  b. $(7 + \sqrt{3}) \cdot (\sqrt{6} - \sqrt{3})$
  c. $7 + \sqrt{3} \cdot (\sqrt{6} - \sqrt{3})$
  d. $(8 - 2\sqrt{3}) \cdot (\sqrt{3} + 4)$

5. Find the missing terms.
  a. $(1 + \sqrt{3})\_\_\_\_ = 3 + \sqrt{3}$
  b. $\sqrt{5} \cdot \_\_\_\_\_ = 10 + 4\sqrt{5}$
  c. $(6 + \sqrt{7})(\_\_\_ + \sqrt{7}) = 55 + 14\sqrt{7}$
  d. $(\sqrt{6} + \sqrt{2}) \cdot \_\_\_\_\_ = 2\sqrt{3} + 2$
  e. $(\sqrt{15} - \sqrt{2}) \cdot \_\_\_\_\_ = 5\sqrt{3} - \sqrt{10}$

DISAPPEARING RADICALS

6. Find the product. Simplify your answer.
  a. $(x - y)(x + y)$
  b. $(x - \sqrt{5})(x + \sqrt{5})$
  c. $(\sqrt{3} - x)(\sqrt{3} + x)$
  d. $(\sqrt{3} - \sqrt{5})(\sqrt{3} + \sqrt{5})$

7. Explain why there are no radicals in the simplified form of any of the answers to problem 6.

8. For each binomial, find a binomial to multiply it by so that the result has no radicals.
   a. $(\sqrt{7} - \sqrt{8})$ b. $(\sqrt{x} + \sqrt{y})$
   c. $(2 - \sqrt{y})$

FRACTIONS AND RADICALS

Definition: To *rationalize* the denominator (or numerator) of a fraction is to write an equivalent fraction with *no radicals* in the denominator (or numerator).

9. Rationalize the denominator. $\frac{1}{2 + \sqrt{3}}$

10. In problem 9, Gerald tried to multiply the numerator and denominator by $(2 + \sqrt{3})$. Explain why this did not work.

11. Daniel used the idea in the section **Disappearing Radicals** to rationalize the denominator. Explain what he did, and why it did work.

12. Rationalize the denominator.
   a. $\frac{1}{\sqrt{2} + 3}$
   b. $\frac{1}{3 - \sqrt{3}}$
   c. $\frac{4}{\sqrt{5} - \sqrt{6}}$
   d. $\frac{5}{\sqrt{5}}$

13. Rationalize the numerator.
   a. $\frac{7 - \sqrt{5}}{4}$ b. $\frac{\sqrt{7} - \sqrt{5}}{4}$

REVIEW CALCULATOR EXPERIMENTS

14. Use your calculator to compute
$$(\sqrt{9876} - \sqrt{9866})(\sqrt{9876} + \sqrt{9866}).$$
Comment on the answer.

15. Bernard believes that the square root of the square of a number is the number itself.
   a. Is he right or wrong? Explain.
   b. What's the square root of the square of -543? Make a prediction, then use your calculator to check.

16. Choose any number. Find its square root on your calculator. Then find the square root of the result. Continue this until you notice something happening. What is happening? Can you explain it? What starting numbers does it work for?

17. Always, sometimes, or never? Explain, using examples.
   a. $x^2 > x$ b. $1/x^2 > 1/x$
   c. $\sqrt{x} < x$ d. $1/\sqrt{x} > 1/x$
   e. $\sqrt{x} < x^2$ f. $1/\sqrt{x} > 1/\sqrt{x^2}$

REVIEW GEOBOARD PUZZLES

18. If two sides of geoboard triangle are $\sqrt{2}$ and $\sqrt{5}$, what are the possibilities for:
   a. the third side?
   b. the area?

19. Find the geoboard figure having the least area, if its perimeter is
   a. 20; b. $4\sqrt{65}$;
   c. $10 + 2\sqrt{65}$; d. $10\sqrt{2} + 2\sqrt{85}$.

LESSON 9.10 Blowups

You will need:

graph paper

BIGGER BOXES

The Real Bag Company makes cardboard boxes. One of the boxes is called the Banker's Box. It has the dimensions: length, 16 in.; width, 12 in.; height, 10 in. Another box, the Square Pak box, has the dimensions: length, 12 in.; width, 12 in.; height, 10 in. Sid, a Real Bag Box Division Manager, decides that new boxes need to be manufactured, the Caterer's Crate and the Great Pak.

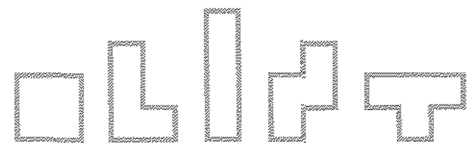
1. **Exploration**

a. The Caterer's Crate will have two dimensions the same as the Banker's Box, and the third dimension multiplied by two. Sid asks his colleague Li Ann whether the volume of the box would be increased the most by multiplying the length, the width, or the height by two. What should she answer? Explain.

b. The Great Pak will have a square base and a volume that is double that of the volume of the Square Pak. Sid asks his colleague Annette (who owns a calculator) to find three choices for the dimensions of the new box. What should she answer? Explain.

STRETCHING POLYOMINOES

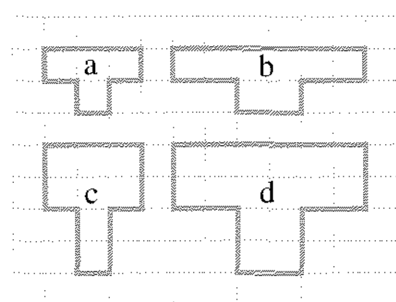
Sid, Annette, and Li Ann like to spend their lunch breaks working geometric puzzles. (They should have become math teachers.) Here is a puzzle they have been working on, using the tetrominoes.



square l i n t

For each of the tetrominoes, they created three new polyominoes. The first one by doubling all horizontal dimensions, the second one by doubling all vertical dimensions, and the third one by doubling both horizontal and vertical dimensions. For example, the t tetromino led to the creation of three new polyominoes.

a. original
b. doubled horizontally
c. doubled vertically
d. doubled both ways



2. **Exploration** Draw all 15 *stretched* tetrominoes. For each one, find its area and perimeter. Keep your work clearly organized, so you can find a pattern to the areas and perimeters. (The area pattern is the easier of the two.) You will need to refer to this data to do the problems in the next two sections.

PERIMETER

Call the perimeter of a tetromino p. It is made up of some horizontal segments and some vertical segments.

Let h = total length of the horizontal segments.
Let v = total length of the vertical segments.

3. Express p in terms of h and v.

4. a. Find h and v for the t tetromino.
b. Show that the perimeter of the vertically stretched t tetromino is $h + 2v$.
c. What is the perimeter of the horizontally stretched t tetromino in terms of h and v?
d. What is the perimeter of the horizontally and vertically stretched t tetromino in terms of h and v?

5. In problem 4 you found formulas that related the perimeters of the three stretched t tetrominoes to the perimeter of the original t tetromino. Explain why these formulas work for all the tetrominoes.

6. What is the sum of the perimeters of the two polyominoes that were stretched in only one dimension? Use factoring to see how this sum is related to the original perimeter.

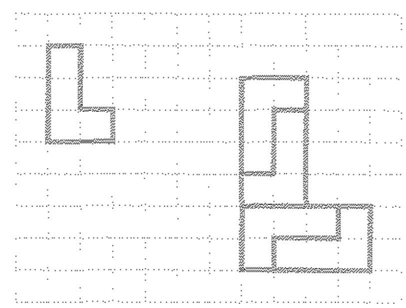
7. Generalization
a. Repeat the perimeter investigation, but stretch the tetrominoes by tripling dimensions. You do not need to draw the tripled tetrominoes, just use algebra. Find a formula relating the perimeters of the tripled tetrominoes to h, v, and p for the original tetromino.
b. Repeat this investigation, but this time stretch by a factor of n.

AREA

8. Refer to your data on the area of the 15 stretched (doubled) tetrominoes, and experiment with other polyominoes. If the original area of a polyomino is A, what is the area of the polyomino stretched by doubling
a. horizontally?
b. vertically?
c. both horizontally and vertically ?

9. a. Draw the l and t tetrominoes, with both their horizontal and vertical dimensions doubled.
b. Repeat part (a), tripling the dimensions instead of doubling.

10. PUZZLE Tile the blown-up tetrominoes you drew with copies of the original l and/or t tetrominoes. Example:



11. How many tetromino tiles did you need to cover the blown-up tetrominoes? How is this related to the area of the blown-up tetrominoes?

12. a. Draw a pentomino.
b. Draw a copy of it, with horizontal and vertical dimensions multiplied by two.
c. Repeat with the original dimensions multiplied by three.
d. Repeat with the original dimensions multiplied by four.

13. Predict the area of each figure you drew in problem 12. Check your predictions.

14. **Generalization** When both horizontal and vertical dimensions are multiplied by k, by what is the area multiplied? Explain.

BACK TO WORK

After their lunch break, Sid, Li Ann, and Annette had to attend to more box problems.

15. They created a new box by multiplying all the dimensions of the Banker's Box by two. Make a sketch of the original box and the new box. What would the volume of the new box be? How many times greater is this than the volume of the Banker's Box?

16. If they created a new box by multiplying all the dimensions of the Square Pak by three, what would its volume be? How many times greater is this than the volume of the Square Pak?

17. **Generalization** When all the dimensions are multiplied by k, by what is the volume multiplied?

18. What are the dimensions of a box that is a perfect cube and has the same volume as the Square Pak? Explain.

19. What are the dimensions of a box that is a perfect cube and has double the volume of the Square Pak? Explain.

REVIEW SCIENTIFIC NOTATION

20. In June of 1990 the national debt of the United States was \$3.1 trillion. The population of the U.S. at the same time was about 250 million. Therefore, the debt per person was

$$\frac{3.1 \text{ trillion}}{250 \text{ million}}.$$

   a. Express both of these numbers in scientific notation.
   b. What was the debt per person? Express your answer in ordinary decimal notation and in scientific notation.

REVIEW WHAT'S YOUR SIGN?

Do not use a calculator for these problems.

21. Is x positive or negative, or is it impossible to know? Explain.
   a. $(-2)^x = -524{,}288$
   b. $2^x = 1/131{,}072$
   c. $(-2)^x = 262{,}144$
   d. $x^{11} = -177{,}147$
   e. $x^{12} = 531{,}441$
   f. $x^{13} = 1/1{,}594{,}323$

REVIEW THE CHESSBOARD

According to an old legend, a King decided to reward the inventor of the game of chess. "I am immensely rich. Whatever you ask for will be yours." The inventor replied, "All I ask is for one cent on the first square of the chessboard; two cents on the next square; four cents on the next square; and so on, doubling the amount each time, until the last square on the chessboard." (The legend actually specifies grains of rice, not cents.)

22. Find out how many cents the King owed the inventor. Express the final answer two ways: in terms of a power of two, in cents; and as a number of dollars, in scientific notation.

23. **Project** Is the money paid the inventor as much as the budget of:

a. a toy store?

b. a multi-national corporation?

c. the State of New York?

d. the United States?

DISCOVERY DECIMAL EXPONENTS

24. Use decimal exponents (to the nearest hundredth) to approximate 100 as a power of:

a. 2	b. 3	c. 4
d. 8	e. 9	f. 10

REVIEW EQUAL RATIOS

Solve for N.

25. $\frac{3N-2}{5} = \frac{N+2}{2}$

26. $\frac{3N-2}{15} = \frac{N+2}{6}$

Solve for x. If you cannot find an exact value, approximate to nearest thousandth.

27. $\frac{x}{8} = \frac{3}{4}$

28. $\frac{4}{10} = \frac{400}{x}$

29. $\frac{1}{x} = \frac{x}{2}$

REVIEW DISTRIBUTIVE LAW PRACTICE

Find these products.

30. $2y(2x - y + 6)$

31. $3x(2x - 3)$

32. $(y - 4)(y + 3)$

LESSON 9.11

Let's Eat!

PIZZA PRICES

Lana and Tina were studying for their semester exam one Sunday afternoon. They needed more energy and decided to order a pizza. They called Pinky's and Primo's to compare prices.

Pinky's Prices

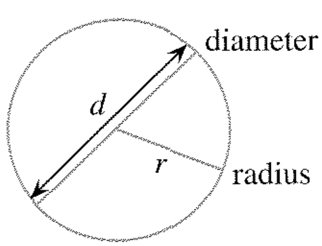
Size	Diameter	Price
small	8 in.	\$4.25
medium	12 in.	\$8.50
large	14 in.	\$10.20

Primo's Prices

Size	Diameter	Price
small	10 in.	\$6.44
medium	12 in.	\$8.84
large	14 in.	\$9.91

1. **Exploration** Assuming the pizzas are of the same thickness and similar quality, which is the better buy for the large pizza? The medium pizza? The small pizza? Explain, showing your calculations.

The area of a circle is given by the formula πr^2, where r is the radius, and π is approximately equal to the number 3.1415926536.



Example: A circle having diameter 14 in. has a radius of 7 in. Its area is $\pi(7)^2$, or 49π square inches.

2. Use your calculator to find the area of a circle having diameter 14 in., to the nearest tenth of a square inch. (Scientific calculators have a π key.)

Tina thought Pinky's medium pizza looked expensive. "It's twice as expensive as an 8-inch pizza," she said. "For twice as much, I ought to be able to get a 16-inch pizza."

3. a. Find the area of a 16-inch pizza. Compare it with the area of an 8-inch pizza. How many times as large is it?

b. How many times as large is a 12-inch pizza as an 8-inch pizza? Show your calculations.

c. Comment on Tina's remark.

4. a. Copy and complete the tables below, giving an approximation for the area of each pizza and the price per square inch.

b. Which pizza is the best buy, based on price per square inch?

Pinky's

Diameter (in.)	Area (sq in.)	Price	Price per sq in.
8	16π	\$4.25	—
12	—	\$8.50	—
14	—	\$10.20	—

Primo's

Diameter (in.)	Area (sq in.)	Price	Price per sq in.
10	—	—	—
12	—	—	—
14	—	—	—

5. Compare the areas of these pizzas. How many times as big is the larger than the smaller?
   a. a 12-inch pizza and a 6-inch pizza
   b. a 14-inch pizza and a 12-inch pizza

6. To compare a pizza having radius r with a pizza having radius $2r$, you can use the ratios of the areas. Simplify this ratio.

$$\frac{\pi(2r)^2}{\pi(r)^2}$$

7. **Generalization** Write and simplify the ratios to compare the area of:
   a. a pizza having radius r with a pizza having three times this radius;
   b. a pizza having radius r with a pizza having radius kr.

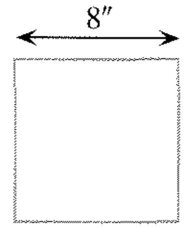
8. If you double the diameter of a pizza, why does the price more than double?

9. For a party Tina was going to buy ten 8-inch pizzas from Pinky's, but she got mixed up and bought eight 10-inch pizzas from Primo's instead. Did she have the right amount of pizza, too much, or too little? Explain, showing your calculations.

BAKING BROWNIES

An hour after they had polished off their pizza, Lana and Tina were having trouble concentrating on studying exponents. "Maybe we're just hungry," said Lana. "I'm feeling a little faint," Tina agreed. "We should probably bake some brownies."

The recipe said to use an 8-inch-square pan.



Lana wanted to double the recipe. "OK," said Tina, "but we'll need a 16-inch-square pan."

10. a. Using the same scale, make a sketch of an 8-by-8-inch pan and a 16-by-16-inch pan.
    b. How many 8-by-8 pans would fit inside a 16-by-16 pan?
    c. Comment on Tina's remark.

11. How many times as big is the larger than the smaller square? (The measurement refers to the side length.)
    a. a 12-inch square and a 6-inch square
    b. a 14-inch square and a 12-inch square

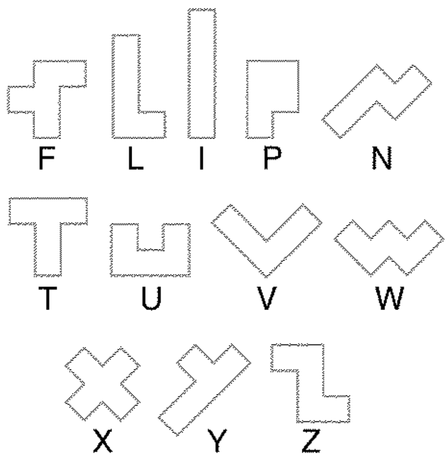
12. **Generalization** What is the ratio of the areas of two squares, if the ratio of the sides is
    a. 5?
    b. k?

13.
    a. Write the ratio of the area of a circle having diameter s to that of a square having side s.
    b. Simplify the ratio. Which is larger, the circle or the square? How many times as large is it?

PUZZLES ***MORE POLYOMINO TILINGS***

14. Draw all the tetrominoes with their dimensions doubled. Tile the blowups with the l and/or t tetrominoes.

15. Repeat with the tripled tetrominoes.



16. Draw all the pentominoes with their dimensions doubled. Tile the blowups with the P and/or N pentominoes.

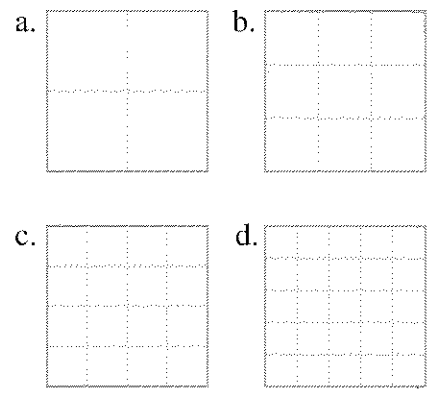
17. Repeat with the tripled pentominoes.

Projects

18. Can you use the same tiles to cover bigger and bigger blown-up tetrominoes and pentominoes? Experiment and report on your discoveries.

19. What is the smallest rectangle you can tile with a given pentomino? Experiment and report on your discoveries.

REVIEW ***A SQUARE NUMBER OF SQUARES***



Each of the four large squares has area 75. Each has been divided into a square number of smaller squares.

20. Find the area of each small square.

21. Express the side of each small square as a square root.

22. Explain why $\sqrt{75} = 5\sqrt{3}$, using
   a. the figure;
   b. radical rules;
   c. decimal approximations.

23. Divide a square having area 72 into a square number of smaller squares, in such a way that you can use the figure to help write $\sqrt{72}$ in simple radical form.

LESSON 9.12

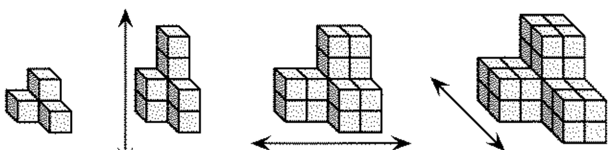
Similar Figures

You will need:

the Lab Gear 

RATIO OF SIMILARITY

The polycubes in this figure were obtained by doubling the dimensions of the original tetracube in succession: first the height, then the length, and finally the width.



a. b. c. d.

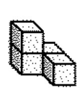
1. Find the volume and surface area of each of these polycubes.

Definitions: Two figures are *similar* if *all* the dimensions of one can be obtained by multiplying the dimensions of the other by the same number, called the *ratio of similarity.*

(In Chapter 3, similar figures were defined as being enlarged or shrunk *without distortion.* That definition is equivalent to this one.)

2. Which two of the four polycubes are similar to each other? Explain.

3. Sketch buildings similar to this tetracube, but larger, with ratio of similarity
   a. 2 b. 3



The two buildings you sketched in problem 3 are similar to each other.

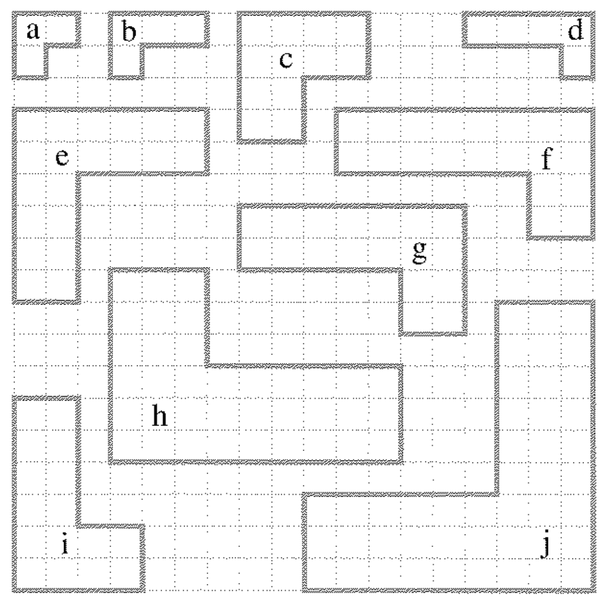
4. You could get the dimensions of the larger building by multiplying the dimensions of the smaller one by what number?

5. You could get the dimensions of the smaller building by multiplying the dimensions of the larger one by what number?

6. Either of the numbers you found in problems 4 and 5 could be considered the ratio of similarity. How are the two numbers related? Explain this.

AREA, PERIMETER

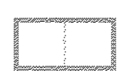
7. Make a list of pairs of similar polyominoes in this figure. (Hint: There are six pairs.)



For each pair of similar polyominoes you found, find

8. the ratio of similarity;

9. the ratio of the areas.



10. Give the dimensions of a rectangle similar to the domino shown above, but larger, such that the ratio of areas is
    a. 25; b. 9;
    c. 2; d. 5.

11. **Generalization**

a. If the ratio of similarity of two figures is R_S, what is the ratio of areas? Explain.

b. If the ratio of areas is R_A, what is the ratio of similarity? Explain.

12. Using the data from problems 7-8, find the relationship between the ratio of similarity and the ratio of perimeters.

13. Make a figure using three 2-D Lab Gear blocks (including some blue blocks).

a. Sketch the figure.

b. Find its perimeter and area.

c. Use blocks to make a figure similar to the original figure.

d. Predict its perimeter and area.

e. Check your prediction.

VOLUME, SURFACE AREA

14. True or False? Explain each one.

a. Any two rectangles are similar.

b. Any two squares are similar.

c. Any two cubes are similar.

15. Build the following cubes using the Lab Gear: 1^3, 5^3, $(x + 1)^3$, and y^3. Find the volume and surface area of each cube.

16. There are six pairs of similar buildings among the four cubes you built. For each pair, find

a. the ratio of similarity;

b. the ratio of surface areas;

c. the ratio of volumes.

17. **Generalization** If you know the ratio of similarity between two figures, R_S, explain how you can find

a. the ratio of surface areas, R_A;

b. the ratio of volumes, R_V.

18. What should be the dimensions of a cubical box that would hold 27 times as much as a box having dimensions 2 in.-by-2 in.-by-2 in.?

19. Repeat problem 18 for a cubical box that would hold 10 times as much as a box having dimensions 2 in.-by-2 in.-by-2 in.

TRAIN SETS

Model train sets come in different scales. The scale is the ratio of similarity between the model and the actual train that is being modeled. This table shows some of the available scales.

Name	Scale
Z	1/220
N	1/160
O	a quarter inch to one foot
HO	an eighth of an inch to one foot
LGB	half an inch to one foot

20. Order the scales from smallest to largest.

21. The LGB scale is also known as 1/25. Comment on this.

George wanted to buy an HO set that would cover an area of 15 square feet.

22. How much area would be covered by the actual train being modeled by this set?

23. How much area would be covered by a similar set in each of the other scales?

24. How many times heavier or lighter do you estimate a similar set would be in each of the other scales? (Assume that you can estimate the ratio of weights by using the ratio of volumes.)

THINKING WRITING

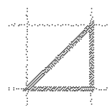
9.C SuperTangrams, Midpoints

SUPERTANGRAMS

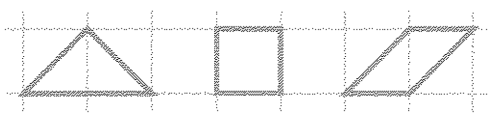
You will need:

- graph paper
- geoboards
- dot paper

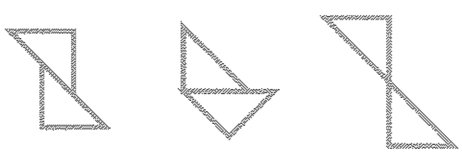
In the world of geometric puzzles, half a unit square (cut along the diagonal), is called a *tan*.



Figures created by combining tans are called *polytans*. Here are the ditans.

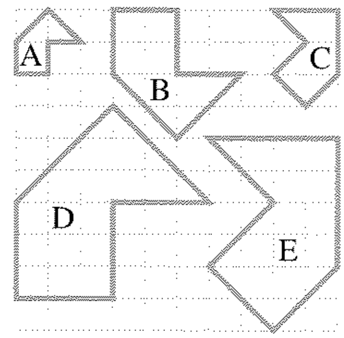


The tans must be combined side-to-side. The following arrangements are not acceptable.



1. Find all four tritans.
2. Find all fourteen tetratans.
3. Tetratans are usually called *SuperTangrams*. Find the perimeter and area of each SuperTangram, using radical expressions when appropriate. Rank the perimeters from shortest to longest.

This figure shows one of the SuperTangrams and four blown-up versions of it.



4. Find the perimeter and the area for the SuperTangram and each blowup.
5. Compare shape A with shape C.
   a. What is the ratio of similarity?
   b. Verify your answer to part (a) by showing that multiplying the perimeter of A by the ratio of similarity yields the perimeter of C.
6. Repeat problem 5 for each other pair of shapes in the figure. (You should find nine more ratios of similarity.)
7. **Report** Write a report summarizing your work in problems 3-6. Include a discussion of:
   - using the Pythagorean theorem;
   - perimeter and area of similar figures;
   - operations with radical expressions.

MIDPOINTS

8. Draw five geoboard segments whose midpoints are on a peg.

9. Make a triangle such that all of its sides have their midpoints on a peg. Connect the midpoints, making a smaller triangle. Study the figure, looking for parallel lines, equal segments, and similar figures.

10. Find the slopes of lines you believe are parallel. Find the lengths of the segments you believe are equal. Find the ratio of similarity for figures you believe are similar.

11. Make a quadrilateral such that all of its sides have their midpoints on a peg. Make the quadrilateral as irregular as you can, avoiding equal or parallel sides. Connect the midpoints, making a smaller quadrilateral. Study the figure, looking for parallel sides and equal segments.

12. Find the slopes of lines you believe are parallel. Find the lengths of the segments you believe are equal.

13. Report Write a report on midpoints of triangles and quadrilaterals. Do you think what you found in the case you investigated will always be true? Explain.

14. Try to make a triangle such that exactly two of its sides have their midpoints on pegs. If you find such a triangle, draw it on dot paper. If you believe such a triangle does not exist, explain why.

Essential Ideas

DISTANCE

1. On the number line, what is the distance between:
a. 12 and -34? b. 12 and 34?
c. 12 and x?

2. On the number line, what points are at distance 7.5 from 6.89?

3. On the number line, what point is halfway between:
a. 12 and -34? b. 12 and 34?
c. 12 and x?

4. (5, 6) is the midpoint of a segment from what point to:
a. (7, -8)? b. (-9.1, 2.34)?

5. What is the biggest possible difference between taxicab and Euclidean distance between two geoboard pegs on a 10-by-10 geoboard? (Give a decimal approximation.)

6. On graph paper, show as many points as possible that are at distance 10 from the origin, using
a. taxicab distance;
b. Euclidean distance.

7. What is the distance from (5, 6) to:
a. (7, -8)? b. (-9.1, 2.34)?

THE PYTHAGOREAN THEOREM

8. How long is the diagonal of a square if the side of the square is
a. 10? b. x?

9. How long is the side of a square if the diagonal is
a. 10? b. x?

10. How long is the other leg of a right triangle, if the first leg is half the hypotenuse, and the hypotenuse is
a. 10? b. x?

FROM ONE POINT TO ANOTHER

11. Given the two points (1, 2.3) and (-4.5, 6), find
a. the taxicab distance between them;
b. the slope of the line that joins them;
c. the Euclidean distance between them.

SQUARE ROOTS

12. Explain why $\sqrt{-4}$ is not a real number.

13. Is $\sqrt{-x}$ a real number? Explain.

14. a. Give three values of x for which $-x$ represents a positive number.
b. Make a table of values and graph $y = \sqrt{-x}$.
c. What is the domain of $y = \sqrt{-x}$?

15. Hal noticed something interesting. He saw that if he squared a number and took its square root, he would get back the same number. Jacob said he could find many numbers for which that wouldn't work. Can you? List some.

16. Ruth thought you could write:
$$-\sqrt{25} = \sqrt{-25} \text{ and } -\sqrt{-25} = \sqrt{25}.$$
Explain why she is wrong.

17. Which is greater? Explain.
a. $\sqrt{80}$ or $8\sqrt{10}$
b. $\sqrt{40} + \sqrt{40}$ or $\sqrt{80}$
c. $\sqrt{63} - \sqrt{28}$ or $\sqrt{63 - 28}$
d. $\frac{\sqrt{4}}{\sqrt{9}}$ or $\sqrt{\frac{4}{9}}$

MULTIPLYING AND DIVIDING

18. What is the area of a rectangle having sides

a. 3 and $\sqrt{6}$?

b. $\sqrt{3}$ and $\sqrt{6}$?

c. $4\sqrt{3}$ and $5\sqrt{6}$?

d. $(4 + \sqrt{3})$ and $5\sqrt{6}$?

19. A rectangle has area $8\sqrt{7}$. Give three possibilities for the sides.

20. A rectangle has area $15 + 6\sqrt{7}$. Give three possibilities for the sides.

21. Write without radicals in the denominator.

a. $\frac{2}{\sqrt{3}}$ b. $\frac{4}{\sqrt{5} + 6}$

ADDING AND SUBTRACTING

22. True or False? Explain.

a. $36 + 64 = 100$

b. $\sqrt{36 + 64} = \sqrt{100}$

c. $\sqrt{36} + \sqrt{64} = \sqrt{100}$

d. $\sqrt{36 + 64} = \sqrt{36} + \sqrt{64}$

23. Simplify, then add or subtract.

a. $\sqrt{8} + \sqrt{72}$

b. $\sqrt{20} - \sqrt{5}$

c. $\sqrt{30} - \sqrt{36} + \sqrt{120} + \sqrt{121}$

d. $15 - \sqrt{15} + 60 - \sqrt{60}$

THE MIDPOINT OF GROWTH

24. Joel invested $200 in 1970 and forgot about it. In the year 2010 he discovered that he had $5227 in the account. How much did he have in the account in 1990 if he was getting

a. simple interest?

b. compound interest?

RADICAL RULES

25. If a and b are nonnegative, write an expression equivalent to each of the following. Explain each rule with an example.

a. $\sqrt{a}\,\sqrt{a}$ b. $\sqrt{a}\sqrt{b}$

c. $a/\sqrt{a}$ d. $\sqrt{a}/\sqrt{b}$

26. Simplify.

a. $\sqrt{2^9}$ b. $\sqrt{2^{10}}$

27. Simplify $\sqrt{2^n}$ assuming n is

a. even; b. odd.

SIMILAR FIGURES

28. Assume you want to use a copy machine to blow up a picture from a 3-inch-by-5-inch index card to 4-inch-by-6-inch card.

a. What percent setting should you use so that you get as large an image as possible, but one which does not extend beyond the edge?

b. How much is the area increased at that setting?

29. Answer the questions in problem 28 about blowing up a picture from a 3-inch-by-5-inch size to an 8.5-inch-by-11-inch size.

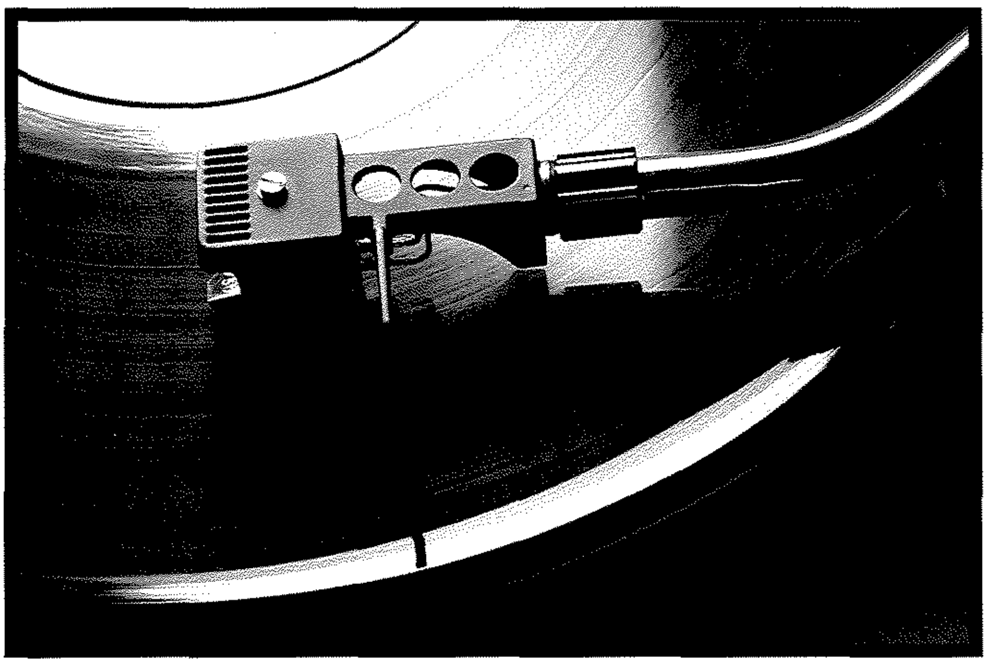
30. Assume you want to use a copy machine to reduce an image so its area gets divided by two. What percent setting should you use?

Assume that the amount of material needed to make clothes is proportional to the surface area, while the amount of food needed is proportional to the volume.

31. How many times as much material would be needed to dress a five-foot Alice as a ten-inch Alice?

32. How many times as much food would be needed to feed a five-foot Alice as a ten-inch Alice?

CHAPTER 10



The spiral groove of a record, from the outer rim to the inner

Coming in this chapter:

Exploration I have pennies, dimes, and quarters, and two bags to put them in. I put all the coins of one kind into one bag, and coins of the other two kinds into the other. There is the same number of coins in each bag, and the total value of each bag's contents is the same. How much money might I have?