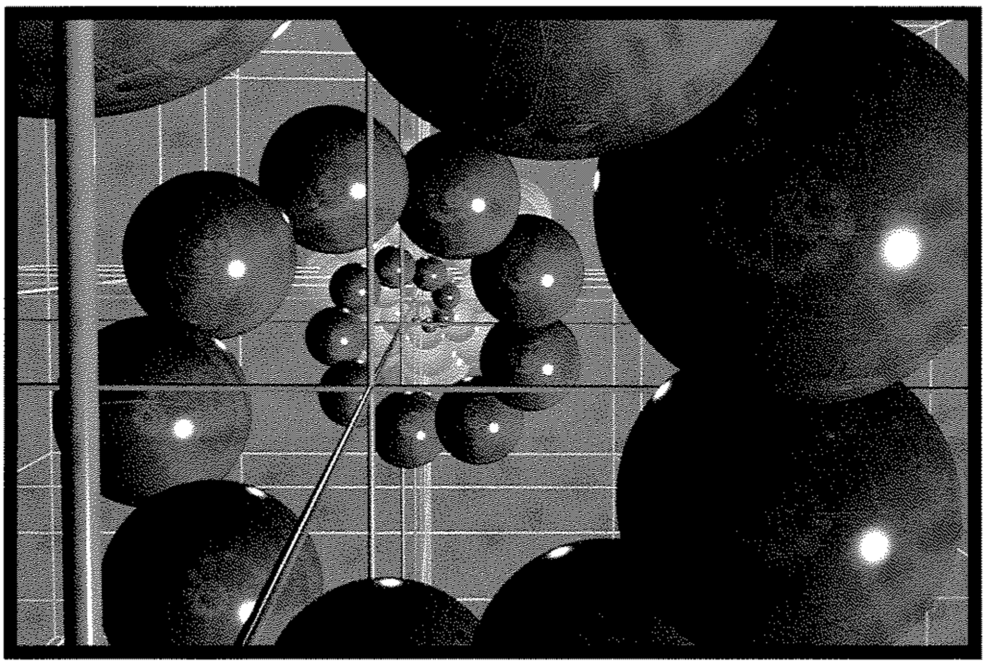
CHAPTER 14



A futuristic spiral

Coming in this chapter:

Exploration Make a paper rectangle that is similar to the smaller rectangle obtained by the following method:

a. cutting the original rectangle in two equal parts;
b. cutting off a square from the original rectangle.

What is the exact ratio of length to width for each of your rectangles?

RATIOS AND ROOTS

LESSON 14.1

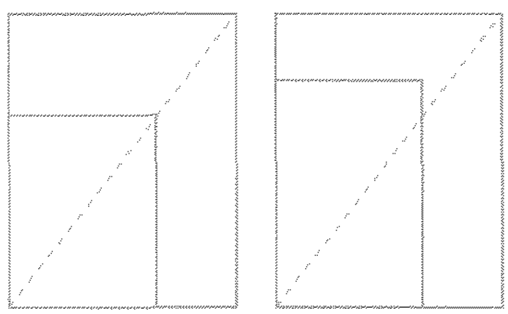
Rectangle Ratios

You will need:

- scissors
- graph paper

HALF RECTANGLES

1. Take two identical rectangular pieces of paper. Fold one in half. Place it on top of the other piece. Is the folded half-rectangle similar to the original rectangle? Check with the diagonal test.



similar not similar

2. **Exploration** Make a paper rectangle, such that the rectangle you get by folding it in half is similar to the original rectangle. What are the dimensions of your rectangle? (Hint: Remember that if two rectangles are similar, their length-to-width ratio must be the same. You may use trial and error on your calculators for different sizes.)

3. a. Sketch a 16-unit-by-12-unit rectangle on graph paper. What is the length-to-width ratio?
   b. Divide the rectangle in half to get one having length 12 and width 8. (The width of the original rectangle becomes the length of the new rectangle.) What is the length-to-width ratio?
   c. Continue to divide the rectangle in half, calculating the ratio of length to width. Make a table like this one to record your data.

Length	Width	*l/w* Ratio
16	12	—
12	8	—

4. Describe any patterns you notice in your table.

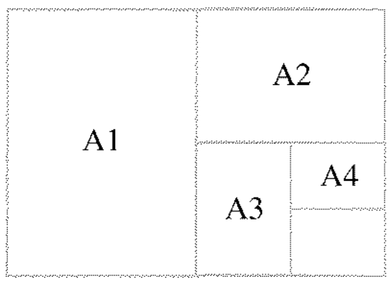
5. a. Repeat problem 2 for three more rectangles. Keep a careful record of your data in tables. Look for patterns.
   b. Find some rectangles for which the length-to-width ratios do not change when you cut them in half.

6. A rectangular sheet of paper is 1 foot wide and x feet long. It is cut into two rectangles, each of which is $(1/2)x$ feet wide and 1 foot long.
   a. Illustrate this in a diagram.
   b. What is the length-to-width ratio in the original rectangle?
   c. What is the length-to-width ratio in each of the two new rectangles?
   d. If the rectangles are similar, we can write an equation setting the original ratio equal to the new ratio. Do this, and find the value of x. Show your calculations.

7. **Report** Summarize your findings from problems 1 through 6. Include sketches and examples. Describe any patterns you noticed. For the rectangles you found in problem 5b, what was the common ratio? What was the common ratio for the rectangle you found in problem 2?

THE INTERNATIONAL PAPER STANDARD

In 1930 an international standard was established for paper sizes, called the *A-series*. The basic size is A0, which is one square meter in area. If you fold it in half, you get paper of size A1. You can fold A1 in half to get A2, fold A2 in half to get A3, etc. The dimensions of A0 were chosen so that *all paper sizes in the series are similar to each other and to A0.*



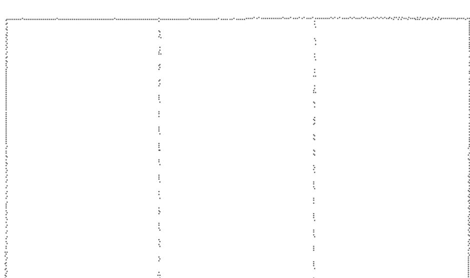
A0

8. Find the dimensions, to the nearest millimeter, of A0 and A1.

DYNAMIC RECTANGLES

The special rectangles you discovered in the previous section each have the property that half of the rectangle is similar to the whole. They are examples of a group of rectangles, called *dynamic rectangles*, that are very useful to artists and designers. Dynamic rectangles have the property that when you cut them into a certain number of equal parts, each of the parts is similar to the whole.

The rectangle below is divided into three parts, each one of which is similar to the original rectangle.



We can express this similarity by writing two equal ratios.

$$\frac{L}{W} = \frac{W}{\frac{1}{3}L}$$

Multiplying both sides of the equation by W:

$$W\left(\frac{L}{W}\right) = W\left(\frac{W}{\frac{1}{3}L}\right)$$

$$L = \frac{W^2}{\frac{1}{3}L}$$

and then by $\frac{1}{3}L$, we get the equation:

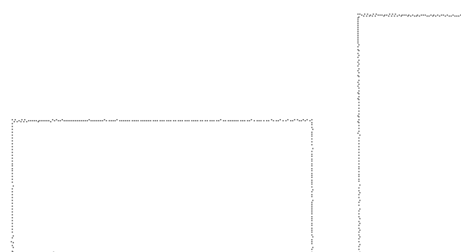
$$\frac{1}{3}L^2 = W^2$$

9. Show how to find L, the length of the original rectangle, if the width is the following:

a. 1 b. 2 c. W

10. What is the ratio of length to width in each of the rectangles in problem 9?

Dynamic rectangles are named for their ratio of length to width. These two rectangles are both called $\sqrt{5}$ rectangles because the ratio of length to width in each of them is $\sqrt{5}$.



11. Into how many equal parts would you divide a $\sqrt{5}$ rectangle in order to make each of the parts similar to the original rectangle? Explain how you figured this out, showing your work.

12. A rectangle is divided into seven parts, each of which is similar to the original rectangle.

a. Give possible dimensions (length and width) for the rectangle.

b. Give another set of possible dimensions.

c. What is the ratio of length to width?

13. A rectangle having width one unit is divided into n equal parts, each of which is similar to the original rectangle.

a. To find the length x of the original rectangle, Tara wrote:

$$\frac{x}{1} = \frac{1}{\frac{1}{n}x}$$

Explain why Tara wrote this proportion.

b. Solve this equation for x.

c. Summarize your results in words.

14. **Research** Many artists and designers use mathematics. Do some research to find out why dynamic rectangles are so useful in art and design. Then make your own design based on dynamic rectangles.

DISCOVERY ***INTERESTING NUMBERS***

15. Find a number that is one more than its reciprocal.

16. Find a number that is one less than its square.

REVIEW ***NUMBERS AND THEIR RECIPROCALS***

If possible, find or estimate the number described. Explain how you found it. (If there is more than one number that fits the description, try to find as many as possible.)

17. The number equals its reciprocal.

18. The number is four more than its reciprocal.

19. The number is one more than twice its reciprocal.

20. The number does not have a reciprocal.

LESSON 14.2

Simplifying Algebraic Fractions

You will need:

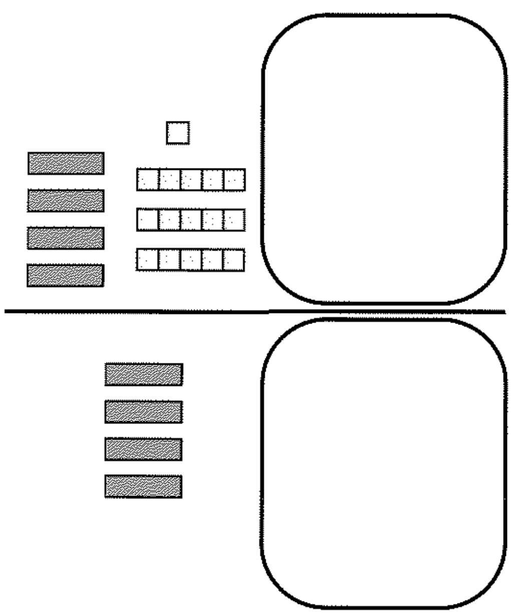
Lab Gear 

ADDING OR SUBTRACTING EQUAL AMOUNTS

1. **Exploration** What happens if you add or subtract equal amounts to or from the numerator and the denominator of a fraction? How can you tell whether the value of the fraction will increase, decrease, or remain the same? Make up several examples to see what happens, then make a generalization.

To model fractions with the Lab Gear, you can use the workmat turned on its side. Instead of representing an equals sign, the straight line in the middle now represents the fraction bar.

Edith and Anna modeled the fraction $\frac{4x + 16}{4x}$ with the Lab Gear, as shown below.



"This is an easy problem," said Edith. "There's a $4x$ in both the numerator and the denominator, so I can get rid of them. The simplified fraction is 16."

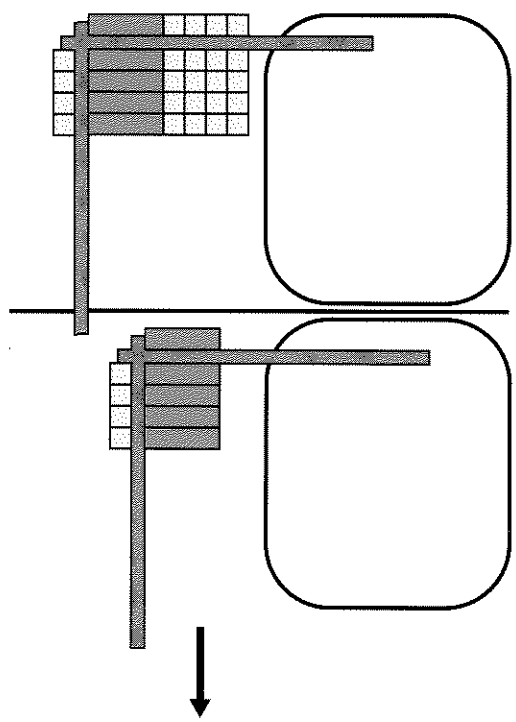
Anna didn't think Edith's method was right. She decided to check Edith's answer by substituting.

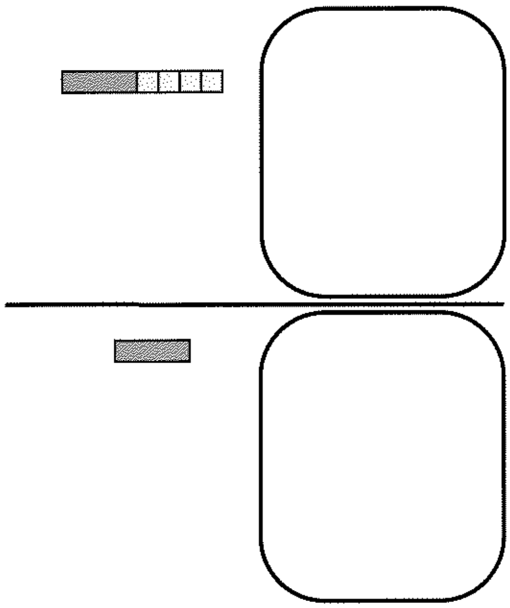
2. Calculate the value of the expression $\frac{4x + 16}{4x}$ for several different values for x. Do all values of x make this fraction equal to 16? Does any value of x make it equal to 16? Explain.

3. Explain why you cannot simplify a fraction by subtracting the same number from the numerator and the denominator. Give examples.

COMMON DIMENSIONS AND DIVISION

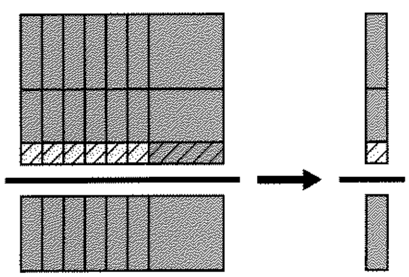
As you know, to simplify a fraction, you *divide numerator and denominator by the same number*. This is still true of algebraic fractions.





4. Study the previous figure.
   a. What are the numerator and the denominator divided by?
   b. What is the simplified fraction?

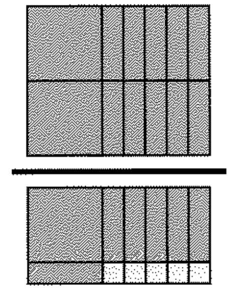
Sometimes, as in the figure below, the numerator and denominator rectangle are seen to have a common dimension, which is the common factor we divide by to get the simplified fraction.



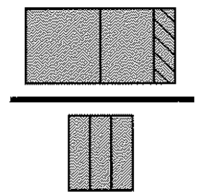
5. Study the preceding figure.
   a. Write the original fraction.
   b. Show what the numerator and denominator must be divided by to simplify the fraction.
   c. Write the simplified fraction.

Repeat problem 5 for the following figures.

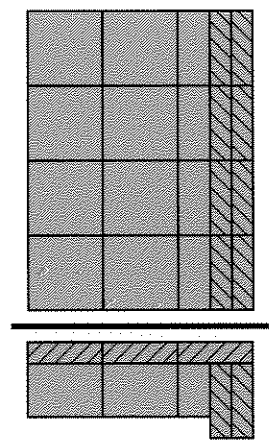
6.



7.



8.



SIMPLIFYING FRACTIONS

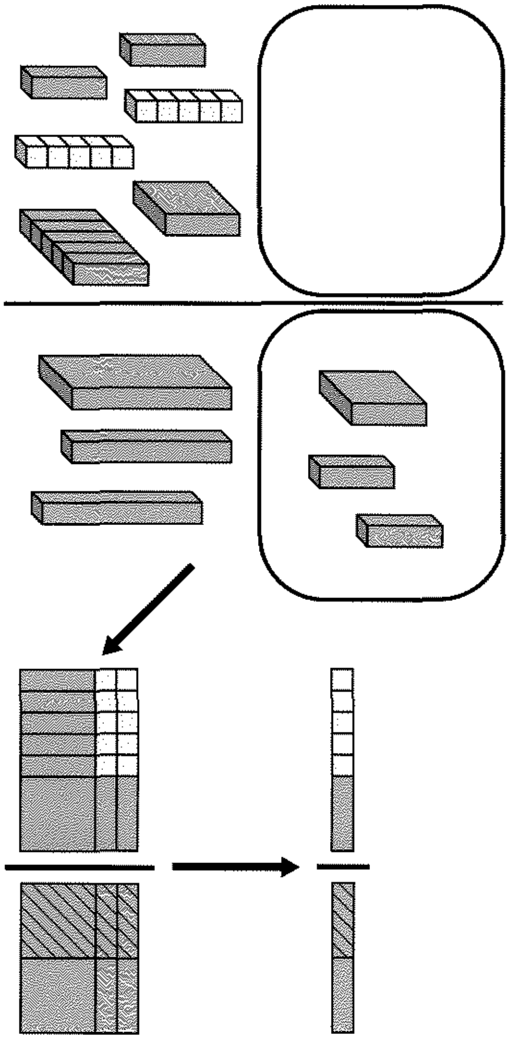
Sometimes it is necessary to factor the numerator and the denominator in order to see the common factors.

Example: Simplify: $\dfrac{x^2 + 3x + 2}{x^2 + 5x + 6}$

Factor: $\dfrac{(x + 2)(x + 1)}{(x + 2)(x + 3)}$

Divide both numerator and denominator by the common factor, $(x + 2)$. The simplified fraction is: $\dfrac{x + 1}{x + 3}$.

The following example is done with the Lab Gear.



9. Explain the process shown in the figure, using words and algebraic notation.

If possible, simplify these fractions.

10. $\frac{3x + 12}{x^2 + 4x}$

11. $\frac{x^2 + 10x + 25}{2x + 10}$

12. $\frac{7x + 5}{7x}$

13. $\frac{2d + 3}{d + 3}$

ZERO IN THE DENOMINATOR

When we substitute 2 for x in the fraction $\frac{3x - 1}{x - 2}$, the denominator has the value zero. Since division by 0 is undefined, we say that the fraction is undefined when $x = 2$.

For what value or values of x (if any) is each fraction undefined?

14. $\frac{2x}{x - 6}$

15. $\frac{x - 6}{x + 6}$

16. $\frac{3}{2x + 6}$

17. $\frac{x^2 + 2}{x^2 - 6x + 8}$

ALWAYS, SOMETIMES, NEVER

Since $\frac{x^2 + 12x + 20}{x + 2}$ can be written

$$\frac{(x + 10)(x + 2)}{x + 2},$$

we can write:

$$\frac{x^2 + 12x + 20}{x + 2} = x + 10$$

18. Explain why the preceding equality is not true when $x = -2$.

19. Explain why it's true when $x \neq -2$.

20. For what value(s) of x is

a. $\frac{2x - 3}{8x - 12} \neq \frac{1}{4}$?

b. $\frac{x^2 - 9}{x - 3} = x + 3$?

Tell whether each equation 21-23 is always true or only sometimes true. If it is only sometimes true, give the values of x for which it is *not* true.

21. $\frac{8x}{4} = 2x$

22. $\frac{x^2 - 1}{8x - 8} = \frac{(x + 1)}{8}$

23. $\frac{5 - 5x}{2x^2 - 2} = \frac{-5}{2x + 2}$

Tell whether each equation 24-26 is always, sometimes, or never true. If it is sometimes true, give the values of x that make it true.

24. $\frac{5x - 5}{5} = 5x$

25. $\frac{5x - 5}{5} = x - 5$

26. $\frac{x^2 - 10}{5} = x^2 - 2$

LESSON 14.3

Fractions and Equations

1. **Exploration** Wanda always enjoyed the math tests Mr. Stevens gave every Friday. She especially liked the tests on fractions. Here is the test she took on Friday the 13th. Try to find the problems she did wrong. If necessary, substitute numbers. If you can, show her how to do them correctly.
   a. $\frac{2x}{5} - \frac{x}{3} = \frac{x}{2}$
   b. $\frac{x}{5} + \frac{x}{5} = \frac{2x}{10}$
   c. $\frac{x}{5} \cdot \frac{x}{5} = \frac{x^2}{25}$
   d. $\frac{2x}{5} \cdot \frac{5}{2x} = 1$
   e. $\frac{x}{5} + \frac{5}{x} = 2$
   f. $\frac{M}{5} = \frac{10M}{50}$
   g. $\frac{2M + 4}{M + 2} = 2$

COMPLICATING FRACTIONS

Sometimes it is useful to *complicate* fractions instead of simplifying them. For example, here are some more complicated fractions that are equivalent to $\frac{2x}{5}$.

a. $\frac{4x^2}{10x}$ b. $\frac{2xy}{5y}$ c. $\frac{8x + 2x^2}{20 + 5x}$

2. What was $\frac{2x}{5}$ multiplied by to give each one of the fractions? Sketch a Lab Gear fraction for part (a).
3. Write three fractions that are equivalent to $2/(x - 3)$. Check the correctness of a classmate's fractions.
4. Write a fraction that is equivalent to $\frac{x + 2}{5}$ that has the following:
   a. a denominator of 10
   b. a denominator of $5x + 15$
   c. a numerator of $4x + 8$
   d. a numerator of $3x^2 + 6x$
5. If possible, write a fraction that is equivalent to $\frac{y + x}{4x}$ that has the following:
   a. a denominator of $8xy$
   b. a denominator of $6x^2$
   c. a numerator of $-2y - 2x$
   d. a numerator of $3y + x$
6. Write a fraction equivalent to 2 that has $5a^2$ as a denominator.
7. Write a fraction equivalent to 1 that has b as a denominator.
8. Write a fraction equivalent to b that has b as a denominator.
9. Write a fraction equivalent to x that has x^2 as a denominator.

COMMON DENOMINATORS

To add or subtract fractions having unlike denominators, you first have to find a common denominator.

10. a. Write a fraction equivalent to $\frac{b}{3}$ having a denominator of $6a^2$.
    b. Add $\frac{b}{3} + \frac{c}{6a^2}$.
11. Write two fractions whose sum is $\frac{2x + 5}{10x}$.
12. a. Write a fraction equivalent to $\frac{bc}{5a}$ having a denominator of $5ac$.
13. Find a common denominator and add or subtract.
    a. $\frac{1}{4x} + \frac{1}{10x^2}$ b. $\frac{5}{xy} - \frac{1}{x^2}$

FROM QUADRATICS TO FRACTIONS

Tara was trying to solve $x^2 + 4x - 6 = 0$ with the zero product property. She couldn't figure out a way to factor the trinomial. Then she had an idea. She wrote:

$$x^2 + 4x = 6$$
$$x(x + 4) = 6$$

Tara was still thinking about the zero product property. She wrote:

$$x = 6 \text{ or } x + 4 = 6$$

14. Explain why Tara's reasoning is incorrect. (Why does this method work when one side of the equation is 0?)

When Tara saw her mistake, she tried another method. She divided both sides by x.

$$x(x + 4) = 6$$
$$x + 4 = \frac{6}{x}$$

Then she was stuck. Her teacher suggested that she use trial and error, so she made this table.

x	$x + 4$	$\frac{6}{x}$
1	5	6
2	6	3
1.5	5.5	4
1.25	5.25	4.8
1.13	5.13	5.31

15. Continue the table and find a value of x that, when substituted into both sides of the equation, will give the same value

a. to the nearest tenth;

b. to the nearest hundredth.

16. The quadratic equation that Tara was solving has two roots. Approximate the other root to the nearest hundredth.

17. Solve the equation $x^2 + 5x - 3 = 0$ using trial and error. (You do not need to do it in the same way as Tara.) Approximate each solution to the nearest hundredth.

18. Confirm your solution by using a method you learned in Chapter 13.

FROM FRACTIONS TO QUADRATICS

Rewrite each equation as an equivalent quadratic equation. Then try to solve it. Show each step.

19. $x + 4 + \frac{3}{x} = 0$

20. $2m + \frac{4}{m} = 9$

21. $4x = \frac{1}{x}$

22. $L - 4 = \frac{20}{L}$

23. $\frac{1}{x} = x + 1$

24. $\frac{4}{x} = x + 1$

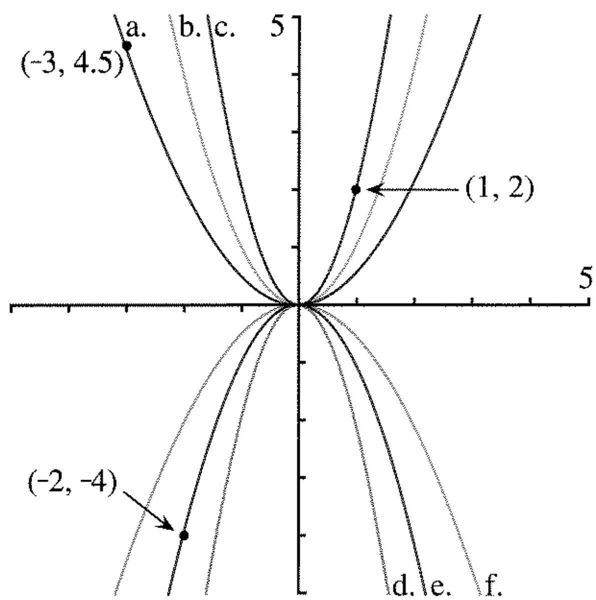
LESSON 14.4 Finding the Vertex

You will need:
- graph paper
- graphing calculators (optional)

Knowing more about quadratic functions and their graphs will help you understand and solve quadratic equations. In particular, it is useful to know how to find the vertex and the x-intercepts of quadratic functions in the following two forms:

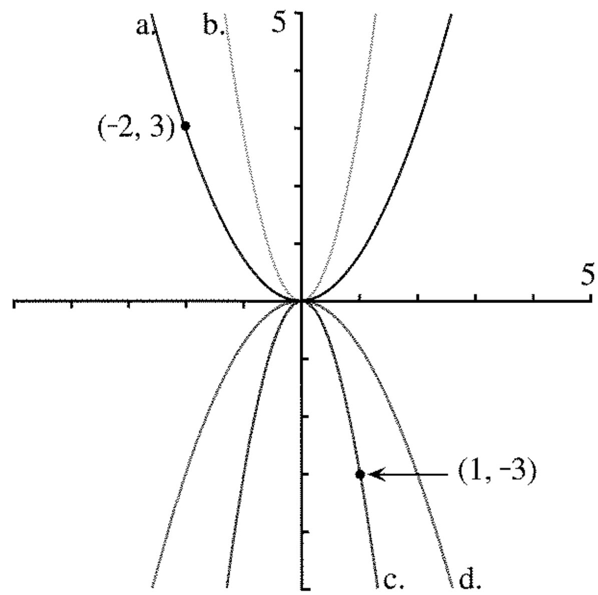
- *Intercept form:* $y = a(x - p)(x - q)$
- *Standard form:* $y = ax^2 + bx + c$

DIFFERENT SHAPES



1. The figure shows several parabolas whose x-intercepts, y-intercept, and vertex are all (0, 0). Match each one with an equation:
   $y = x^2$ $\quad y = 0.5x^2 \quad y = 2x^2$
   $y = -x^2 \quad y = -0.5x^2 \quad y = -2x^2$

2. What is the value of a for the parabolas on the following figure?



3. Which among the parabolas in problems 1 and 2 is most open? Most closed? How is this related to the value of a?

4. Write the equation of a parabola that lies entirely between parabolas 1a and 1b.

5. Describe the graph of:
   a. $y = -0.01x^2$; b. $y = 100x^2$.

6. **Summary** Explain the effect of the parameter a, in the function $y = ax^2$, on the shape and orientation of the graph.

INTERCEPT FORM

As you learned in Chapter 13, when the equation is in intercept form, you can find the vertex from the x-intercepts, which are easy to locate.

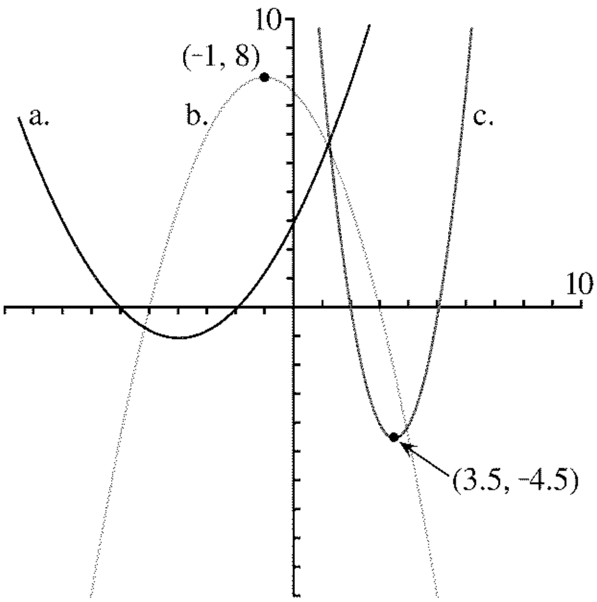
7. Try to answer the following questions about the graph of $y = 2(x - 3)(x + 4)$ without graphing.
   a. What are the x- and y-intercepts?
   b. What are the coordinates of the vertex?

8. **Generalization**

a. What are the x- and y-intercepts of $y = a(x - p)(x - q)$? Explain.

b. Explain in words how to find the vertex if you know the intercepts.

9. The figure shows the graphs of several parabolas. Write an equation for each one. (Hint: To find a, use either the y-intercept or the vertex and algebra or trial and error.)



10. For each equation, tell whether its graph is a smile or a frown parabola, without graphing. Explain your reasoning.

a. $y = 9(x - 8)(x - 7)$

b. $y = -9(x - 8)(x - 7)$

c. $y = 9(8 - x)(x - 7)$

d. $y = 9(8 - x)(7 - x)$

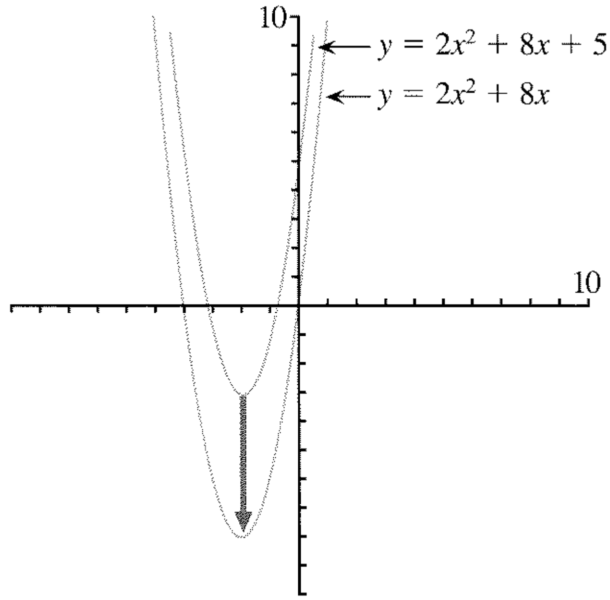
11. If you know all the intercepts and the vertex of $y = 3(x - p)(x - q)$, explain how you would find the intercepts and the vertex of $y = -3(x - p)(x - q)$.

STANDARD FORM

When the equation is in standard form, $y = ax^2 + bx + c$, it is more difficult to find the location of the vertex. One particularly easy case, however, is the case where $c = 0$.

12. Explain why when $c = 0$, the parabola goes through the origin.

13. Find the vertex of $y = 2x^2 + 8x$. (Hint: Factor to get into intercept form.)



14. How are the two graphs related? Compare the axis of symmetry and the y-intercept.

15. How is the graph of $y = 2x^2 + 8x - 3$ related to them?

16. Find the equation of any other parabola whose vertex is directly above or below the vertex of $y = 2x^2 + 8x$.

FINDING *H* AND *V*

Example: Find the coordinates (H, V) of the vertex of the graph of $y = 3x^2 - 18x + 7$.

- $y = 3x^2 - 18x$ is the vertical translation for which $V = 0$. By factoring, we see it is equal to $y = x(3x - 18)$.
- To find the x-intercepts of $y = 3x^2 - 18x$, we set $y = 0$. By the zero product property, one x-intercept is 0. To find the other, we solve the equation $3x - 18 = 0$, and get $x = 6$.
- Since the x-intercepts are 0 and 6, and the axis of symmetry for both parabolas is halfway between, it must be 3. So $H = 3$.

- Substitute 3 into the original equation to see that the y-coordinate of the vertex is:

  $V = 3(3)^2 - 18(3) + 7 = -20.$

  So the coordinates of the vertex for the original parabola are (3, -20).

17. For each equation, find H and V. It may help to sketch the vertical translation of the parabola for which $V = 0$.

a. $y = x^2 + 6x + 5$
b. $y = 2x^2 + 6x + 5$
c. $y = 3x^2 - 6x + 5$
d. $y = 6x^2 - 6x + 5$

Generalizations

18. What is the equation of a parabola through the origin that is a vertical translation of $y = ax^2 + bx + c$?

19. Show how to find the axis of symmetry of:

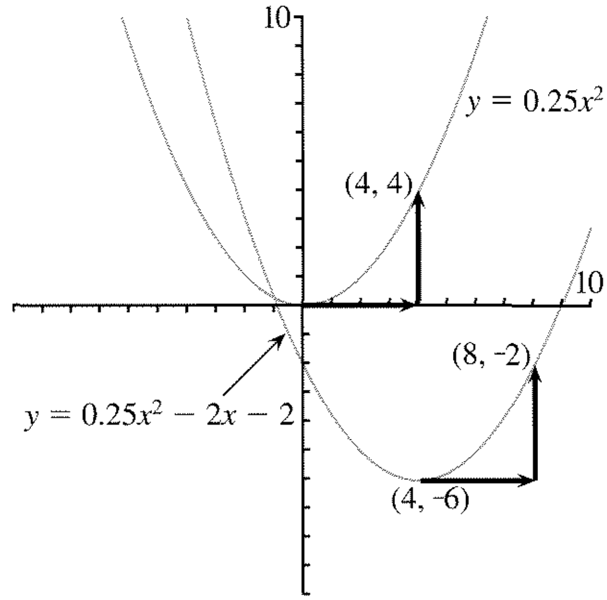
a. $y = ax^2 + bx$;
b. $y = ax^2 - bx$.

20. Explain why the x-coordinate of the vertex of the parabola having equation $y = ax^2 + bx + c$ is

$$H = -\frac{b}{2a}.$$

SAME SHAPE

The parameter a determines the shape of the parabola. The graphs of all equations in standard form that share the same value for a are translations of the graph of $y = ax^2$.



For example, the two parabolas in the figure have equations with $a = 0.25$. Therefore they have the same shape, as the following exercise shows.

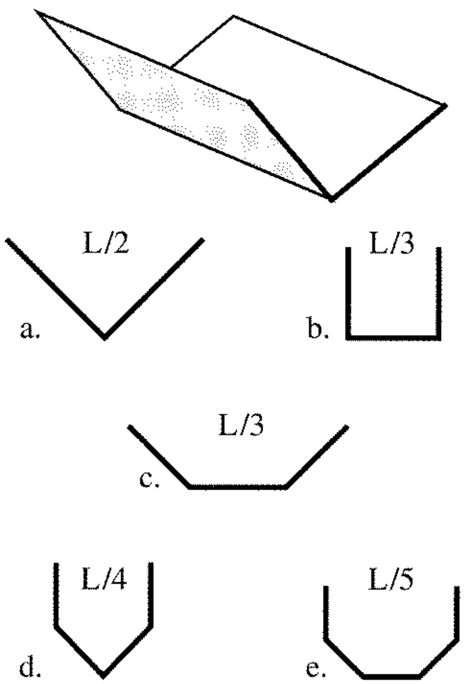
21.

a. Show algebraically that starting at the vertex, and moving 4 across and 4 up, lands you on a point that satisfies the equation in both cases.
b. If you move 2 across from the vertex, show that you move up the same amount to get to the parabola in both cases.

THINKING WRITING

14.A In the Gutter

You have a long rectangular sheet of metal, having width L inches. You intend to fold it to make a gutter. You want to find out which of the folds shown in the figure will give the maximum flow of water. This depends on the area of the cross-section of the gutter; a bigger area means better flow.



1. Find the area of the cross-section for the examples shown in the figure. (All angles are 90 or 135 degrees. All sides in each cross-section are of equal length. Hint: Divide the areas into rectangles and right triangles that are half-squares.) Which cross-section has the greatest area?

2. You may try the same shapes with different dimensions. For example, for cross-section b, you could have a height of $L/4$, and a width of $L/2$. Try to increase the areas for cross-sections b, c, d, and e by choosing different values for the different segments. (Remember that the sum of all the lengths must be L.)

3. **Report** Figure out the best design for a gutter. Write an illustrated report on your research, explaining clearly how you arrived at your conclusions. You need not limit yourself to the shapes given here.

LESSON 14.5

A Famous Formula

You will need:

- graph paper
- graphing calculators (optional)

STANDARD FORM OF A QUADRATIC

Definition: A quadratic equation is said to be in *standard form* if it is in the form:

$$ax^2 + bx + c = 0.$$

In Chapter 13 you learned several methods to solve quadratics in the case where $a = 1$. If you divide all the terms of a quadratic equation in standard form by a, you can solve it with those methods.

Example: Solve $3x^2 + 5x - 4 = 0$.
Divide both sides by 3:

$$x^2 + \frac{5}{3}x - \frac{4}{3} = \frac{0}{3}$$

$$x^2 + \frac{5}{3}x - \frac{4}{3} = 0.$$

Since $a = 1$, the solutions are $H \pm \sqrt{-V}$. In this case:

$$H = -b/2 = -5/6.$$

Find V by substituting H for x in the equation.

$$\begin{aligned} V &= \left(\frac{-5}{6}\right)^2 + \left(\frac{5}{3}\right)\left(\frac{-5}{6}\right) - \frac{4}{3} \\ &= \frac{25}{36} - \frac{25}{18} - \frac{4}{3} \\ &= \frac{25}{36} - \frac{50}{36} - \frac{48}{36} \\ &= \frac{-73}{36} \end{aligned}$$

So the solutions are:

$$-\frac{5}{6} + \sqrt{\frac{73}{36}} \text{ or } -\frac{5}{6} - \sqrt{\frac{73}{36}}$$

The two solutions can be written as one expression:

$$-\frac{5}{6} \pm \sqrt{\frac{73}{36}}$$

where the symbol $\pm$ is read *plus or minus*. It is also possible to write it as a single fraction:

$$-\frac{5}{6} \pm \sqrt{\frac{73}{36}} = -\frac{5}{6} \pm \frac{\sqrt{73}}{6} = \frac{-5 \pm \sqrt{73}}{6}$$

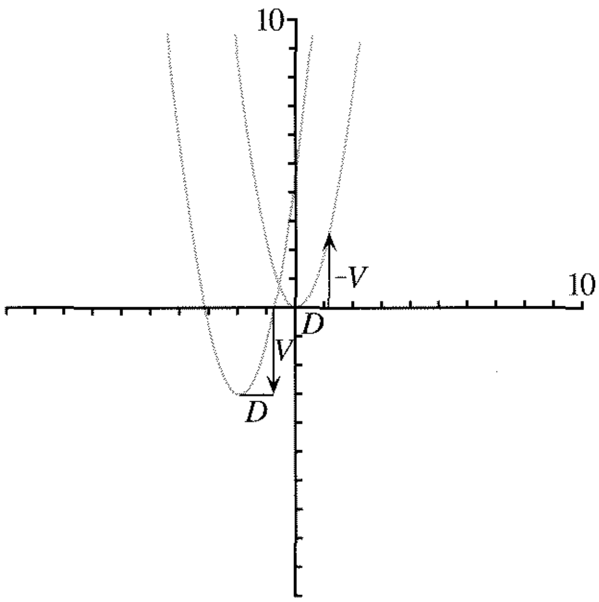
Solve. (Hint: You may divide by a, then use any of the methods from Chapter 13.)

1. $2x^2 + 4x - 8 = 0$
2. $-x^2 + 4x + 8 = 0$
3. $3x^2 + 4x - 4 = 0$
4. $-3x^2 + 8x + 8 = 0$

FINDING THE x-INTERCEPTS

You already know how to find the vertex of a quadratic function in standard form. In this section you will learn how to find the x-intercepts from the vertex.

The following figure shows the graph of the function $y = ax^2 + bx + c$, which is a translation of $y = ax^2$, whose graph is also shown. The coordinates of the vertex are (H, V). D is the distance from the x-intercepts to the axis of symmetry. When $a = 1$, we found that $D = \sqrt{-V}$. What is D in the general case?



The figure shows D and V on a parabola that was translated from $y = ax^2$. In this example, V was a negative number, and the translation was in a downward direction. The arrows representing D and V are also shown on the original parabola. (On $y = x^2$, the direction of the arrow for V was reversed. What is shown is actually the opposite of V. This is indicated by the label $-V$. Since V is negative, $-V$ is positive.)

5. Use the figure to explain why $-V = aD^2$.

6. Express D in terms of V and a.

7. This formula is different from the one we had found in the case where $a = 1$. Explain why this formula works whether $a = 1$ or $a \neq 1$.

SOLVING QUADRATIC EQUATIONS

The x-intercepts, when they exist, are equal to $H \pm D$. It follows from the value of D found in the previous section that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by the formula:

$$H \pm \sqrt{-\frac{V}{a}}.$$

Therefore, one way to solve a quadratic equation in standard form is first to find H and V. In Lesson 2 you learned how to express H in terms of a and b. Then V can be found by substituting H into the equation.

Example: Solve $2x^2 + 8x - 7 = 0$.

Solutions:

$$H \pm \sqrt{-\frac{V}{a}} = -2 \pm \sqrt{-\frac{-15}{2}} = -2 \pm \sqrt{7.5}$$

Solve.

8. $2x^2 + 6x - 8 = 0$

9. $-x^2 + 6x + 8 = 0$

10. $3x^2 + 6x + 1 = 0$

11. $-3x^2 + 6x + 8 = 0$

THE QUADRATIC FORMULA

As you know, $H = -b/(2a)$. The following problem uses that fact to find a formula for V in terms of a, b, and c.

12. Substitute $-b/(2a)$ into $ax^2 + bx + c$ to find the y-coordinate of the vertex as a single fraction in terms of a, b, and c.

If you did problem 12 correctly, you should have found that:

$$V = \frac{-b^2 + 4ac}{4a}.$$

13. To find a formula for the solutions of the quadratic equation in standard form in terms of a, b, and c, substitute the expressions for H and V into the expression

$$H \pm \sqrt{-\frac{V}{a}}.$$

If you did this correctly, you should have found that the solutions are:

$$-\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

14. Show that this simplifies to:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This expression is the famous *quadratic formula.* It gives the solutions to a quadratic equation in standard form in terms of a, b, and c. You will find it useful to memorize it as follows: "The opposite of b, plus or minus the square root of b squared minus $4ac$, all over $2a$."

Solve these equations. (If you use the quadratic formula, you are less likely to make mistakes if you calculate the quantity $b^2 - 4ac$ first.)

15. $2x^2 + 6x - 4 = 0$

16. $-x^2 + 6x + 4 = 0$

17. $3x^2 + 6x - 4 = 0$

18. $-3x^2 + 7x - 4 = 0$

19. Report What are all the methods you know for solving quadratic equations? Use examples.

DISCOVERY *A TOUGH INEQUALITY*

On Friday night when Mary and Martin walked into the G. Ale Bar, Ginger gave them a challenging inequality. "This stumps some calculus students," she said, "but I think you can figure it out."

20. Solve Ginger's inequality: $3 < 1/x$. Check and explain your solution.

REVIEW *RECTANGLES*

21. The length of a rectangle is 10 more than the width. Write a formula for:

a. the width in terms of the length;

b. the area in terms of the length;

c. the perimeter in terms of the width.

22. A rectangle has width $3x + 1$ and length $6x + 2$. Find the perimeter when the area is 200.

LESSON 14.6

Translations of $y = ax^2$

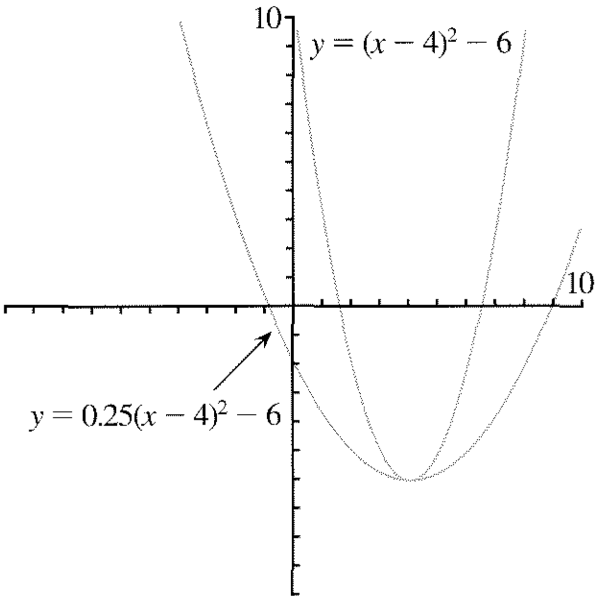
You will need:

graph paper

graphing calculators (optional)

VERTEX FORM

In Chapter 13 you learned that the parameters H and V in the equation $y = (x - H)^2 + V$ represent the coordinates of the vertex of a parabola which is a translation of the one with equation $y = x^2$. This is easy to generalize to any equation in the form $y = a(x - H)^2 + V$, even when $a \neq 1$.



The two parabolas shown in the figure have the same vertex.

1. Write the equation of a parabola having the same vertex as both in the figure that is
   a. more open than either;
   b. more closed than either;
   c. between the two.

2.
   a. Explain why the lowest value for the quantity $(x - 4)^2$ is 0.
   b. Explain how it follows that the lowest point for both parabolas must be for $x = 4$.

3. Write the equation of the parabola that has the same shape as $y = 0.25x^2$ having vertex (-3, 2).

4. Find the equation of a parabola that is a translation of $y = 5x^2$ having vertex (4, -2).

5. The following questions are about the function $y = 6(x + 5)^2 - 4$.
   a. What are the coordinates of the vertex of its graph?
   b. What is the equation of the parabola of the same shape having the vertex at the origin?
   c. What is the equation of the frown parabola having the same shape, and the vertex at the origin?
   d. What is the equation of the frown parabola having the same shape and vertex?

6. **Summary** What do you know about the shape and vertex of the graph of $y = a(x - H)^2 + V$?

MORE ON EQUAL SQUARES

Use the equal squares method to solve each equation. Notice how the solutions of the first equation in each pair differ from the solutions of the second equation.

7. a. $x^2 - 9 = 0$
   b. $4x^2 - 9 = 0$

8. a. $x^2 - 6 = 0$
b. $9x^2 - 6 = 0$

9. a. $(x - 3)^2 - 5 = 0$
b. $16(x - 3)^2 - 5 = 0$

10. a. $(x + 2)^2 - 7 = 0$
b. $3(x + 2)^2 - 7 = 0$

11. **Generalization**

a. Describe how the roots of the second equation in each pair differ from the roots of the first equation.

b. Use the equal squares method to find a general formula for the solutions of the equation $a(x - H)^2 + V = 0$. Explain.

If you did problem 11 correctly, you should have found the same formula as in Lesson 5.

$$H \pm \sqrt{-\frac{V}{a}}$$

COMPLETING THE SQUARE

You can change a quadratic equation from standard form to vertex form by completing the square. When $a \neq 1$, it is more difficult, but it can still be done.

Example: Write $y = 3x^2 + 6x - 9$ in vertex form.

Then complete the square for the quantity inside the parentheses:

$$y = 3(x^2 + 2x + 1 - 1 - 3)$$

Finally, distribute the 3:

$$y = 3(x + 1)^2 - 12$$

So $H = -1$ and $V = -12$. You can check that this was done correctly by finding H and V using the method from Lesson 4:

$$V = 3(-1)^2 + 6(-1) - 9 = -12$$

The same method for completing the square is used even when a is not a common factor.

Example: Write $y = 3x^2 + 5x - 7$

Factor the 3:

$$y = 3\left(x^2 + \frac{5}{3}x - \frac{7}{3}\right)$$

Complete the square:

$$y = 3\left(x^2 + \frac{5}{3}x + \frac{25}{36} - \frac{25}{36} - \frac{7}{3}\right)$$

$$= 3\left(\left(x + \frac{5}{6}\right)^2 - \frac{109}{36}\right)$$

Distribute the 3:

So $H = -5/6$ and $V = -109/12$.

12. Check that H and V were found correctly.

Complete the square.

13. $y = 3x^2 + 6x + 9$

14. $y = -2x^2 + 5x + 8$

15. $y = 2x^2 - 5x + 3$

THE QUADRATIC FORMULA, AGAIN

Let us write $y = ax^2 + bx + c$ in vertex form by completing the square.

Factor the a:

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Complete the square:

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a^2}\right)$$

Distribute the a:

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a}$$

So $H = \frac{-b}{2a}$, and $V = \frac{-b^2 + 4ac}{4a}$ as we saw in Lesson 5.

Finally, if we solve the equation

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a} = 0$$

by the equal squares method, we get:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

So:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

DISCOVERY **EGYPTIAN FRACTIONS**

The ancient Egyptians used only those fractions having 1 for the numerator.

16. Find the sum. Look for patterns.

a. $\frac{1}{5} + \frac{1}{20} = \frac{1}{?}$ b. $\frac{1}{3} + \frac{1}{6} = \frac{1}{?}$

c. $\frac{1}{4} + \frac{1}{12} = \frac{1}{?}$

17. Use the above pattern to predict these missing denominators.

a. $\frac{1}{7} + \frac{1}{?} = \frac{1}{6}$ b. $\frac{1}{?} + \frac{1}{30} = \frac{1}{5}$

c. $\frac{1}{10} + \frac{1}{90} = \frac{1}{?}$

18. Write three more problems having the same pattern as above.

19. Generalization

a. Write an algebraic statement to describe the pattern you found in #16. Use expressions in terms of D for m and n in the equality.

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{D}$$

b. Use algebra to check that your statement is an identity.

20. Find x. Look for patterns.

a. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{x} + \frac{1}{x}$

b. $\frac{1}{4} + \frac{1}{5} + \frac{1}{20} = \frac{1}{x} + \frac{1}{x}$

21. Use the above pattern to express the following fractions as a sum of Egyptian fractions. Check your answers.

a. $\frac{2}{5}$ b. $\frac{2}{7}$

22. Generalization

a. Write an algebraic statement to describe the pattern.

b. Use algebra to check that your statement is an identity.

LESSON 14.7

Equations and Numbers

In this lesson we will discuss quadratic functions and equations in standard form, $y = a^2 + bx + c$ and $ax^2 + bx + c = 0$.

HOW MANY x-INTERCEPTS?

A quadratic equation may have 2, 1, or 0 real number solutions, depending on the number of x-intercepts on the graph of the corresponding function.

1. Sketch a parabola for whose equation:

a. $a > 0$ and $c < 0$

b. $a < 0$ and $c > 0$

2. Explain why a parabola for which a and c have opposite signs must intersect the x-axis.

3. Sketch a parabola to explain why if $a > 0$ and $V < 0$ there are two x-intercepts.

4. Fill the table with the number of x-intercepts for a quadratic function with the given signs for a and V. Justify each answer with a sketch.

	$V < 0$	$V = 0$	$V > 0$
$a > 0$	2	—	—
$a < 0$	—	—	—

(We do not consider the case $a = 0$, since then the function is no longer quadratic.)

5. How many x-intercepts are there if:

a. $V = 0$?

b. V and a have the same sign?

c. V and a have opposite signs?

In Lesson 6 you found that $V = \dfrac{-b^2 + 4ac}{4a}$.

Definition: The quantity $b^2 - 4ac$, which appears under the radical in the quadratic formula, is called the *discriminant*, which is sometimes written Δ (the Greek letter *delta*).

6. Explain why we can write $V = -\Delta/(4a)$.

HOW MANY SOLUTIONS?

It turns out that the discriminant allows us to know the number of solutions of a quadratic equation. Refer to the table in problem 4 to answer the following questions.

7. If $\Delta = 0$, what is V? How many solutions are there?

8. If $\Delta > 0$,

a. and $a > 0$, what is the sign of V? How many solutions are there?

b. and $a < 0$, what is the sign of V? How many solutions are there?

9. If $\Delta < 0$,

a. and $a > 0$, what is the sign of V? How many solutions are there?

b. and $a < 0$, what is the sign of V? How many solutions are there?

The quadratic formula can be written:

$$\frac{-b \pm \sqrt{\Delta}}{2a}$$

10. Summary Using the quadratic formula, explain why,

a. if $\Delta = 0$ there is only one solution;

b. if $\Delta < 0$ there are no real solutions;

c. if $\Delta > 0$ there are two real solutions.

11. Explain why if a and c have opposite signs, the discriminant cannot be negative.

SUM AND PRODUCT OF THE SOLUTIONS

12. In the case where there are two solutions $x_1 = \frac{-b + \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b - \sqrt{\Delta}}{2a}$,

a. what is $x_1 + x_2$?

b. what is the average of x_1 and x_2? (How is this related to the axis of symmetry?)

c. what is $x_1 \cdot x_2$?

The sum of the solutions of a quadratic equation is $S = -b/a$, and the product is $P = c/a$. This provides a quick way to check the correctness of the solutions to a quadratic.

Example: Phred solved the quadratic equation $2x^2 + 5x - 8 = 0$ and got $\frac{-5 \pm \sqrt{89}}{2}$. To check the correctness of the answer, he added the two roots, hoping to get $S = -b/a = -5/2$. Conveniently, the $\sqrt{89}$ disappeared:

$$\frac{-5 + \sqrt{89}}{2} + \frac{-5 - \sqrt{89}}{2} = \frac{-10}{2}$$

Since $-10/2 \neq -5/2$, Phred must have made a mistake.

Solve, and check the correctness of your answers, with the help of S and P (or by substituting in the original equation).

13. $2x^2 + 5x - 8 = 0$

14. $2x^2 - 8x + 5 = 0$

15. $-8x^2 + 3x + 5 = 0$

16. $-2x^2 - 5x - 1 = 0$

KINDS OF NUMBERS

The first numbers people used were whole numbers. It took many centuries to discover more and more types of numbers. The discovery of new kinds of numbers is related to the attempt to solve more and more equations. The following equations are examples.

a. $x + 2 = 9$ b. $x + 9 = 2$

c. $2x = 6$ d. $6x = 2$

e. $x^2 = 9$ f. $x^2 = 10$

g. $x^2 = -9$

17. Pretend you know about only the *natural numbers*. (These are the positive whole numbers.) List the equations a-f that can be solved.

18. Pretend you know about only the *integers*. (These are positive and negative whole numbers and zero.) List the equations a-f that can be solved. Find one that has two solutions.

19. Pretend you know about only the *rational numbers*. (These are all fractions, positive, negative, and zero. Of course, integers are included, since for example $3 = 6/2$.) List the equations a-f that can be solved.

20. The *real numbers* include all rational and irrational numbers. Which equations can be solved if you know about all the real numbers?

Natural numbers, integers, rational numbers, and real numbers can all be found on a one-dimensional number line. However, to solve equation (g), you need to get off the number line. The solution is a *complex number*, and it is written $3i$. The number i is a number one unit away from 0, but off the number line. It is defined as a number whose square is -1:

$$i^2 = -1.$$

Complex numbers cannot be shown on a line. They require a two-dimensional number plane. You will learn more about them in future math classes.

21. Create an equation whose solution is

a. a natural number;

b. an integer, but not a natural number;

c. a rational number, but not an integer;

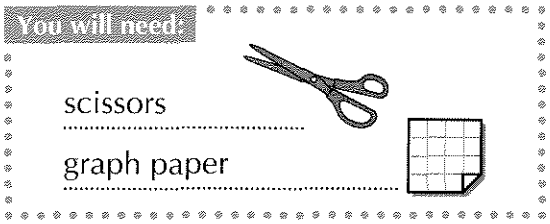
d. an irrational number.

22. Create an equation that has no real number solution.

23. **Research** Find out about complex numbers.

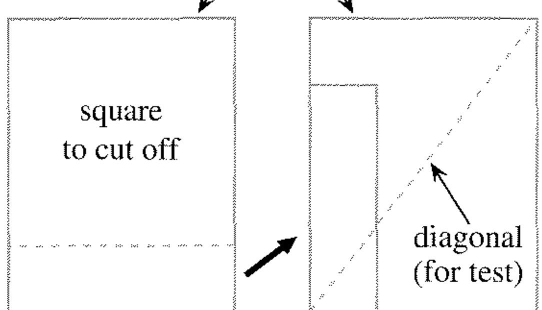
LESSON 14.8

The Golden Ratio



THE GOLDEN RECTANGLE

1. Take two identical rectangular pieces of paper. Cut a square off one end of one of them, as shown in the figure. Is the remaining rectangle similar to the original one? Check with the diagonal test.



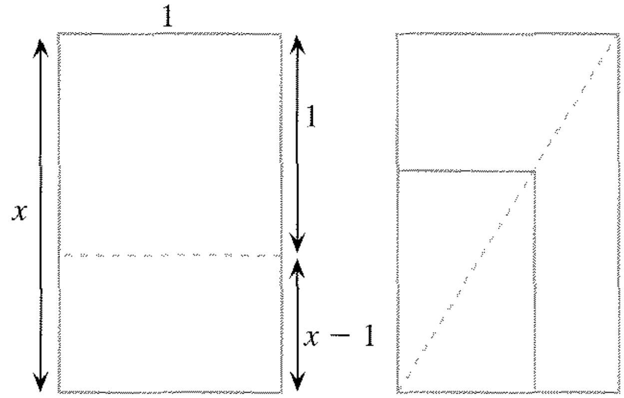
2. **Exploration** Make a paper rectangle, such that the rectangle that remains after cutting off a square is similar to the original rectangle. What are the dimensions of your rectangle? (Hint: Remember that if two rectangles are similar, their length-to-width ratio must be the same. You may use trial and error on your calculators for different sizes, or write and solve equations.)

Definitions:

- A *golden rectangle* is one that satisfies the following property: If you cut a square off one end of the rectangle, the remaining rectangle is similar to the original one.
- The ratio of the longer to the shorter side of a golden rectangle is called the *golden ratio*.

Golden rectangles and the golden ratio are used frequently in art, design, and architecture.

3. What is the length-to-width ratio of the rectangle you found in problem 2? Compare your answer with your classmates' answers.



This figure shows a golden rectangle (on the left). To find the exact value of the golden ratio, we will write and solve an equation about the similar rectangles shown (on the right).

4. Explain why $\frac{x-1}{1} = \frac{1}{x}$.

5. Solve the equation.

There should be two solutions. The positive one is the golden ratio.

6. What is the exact value of the golden ratio?

7. What is the golden ratio, rounded to the nearest one thousandth?

8. What is the reciprocal of the golden ratio, rounded to the nearest one thousandth?

Notation: The golden ratio is often represented by the Greek letter φ (*phi*).

A SPECIAL SEQUENCE

A Fibonacci-like sequence is one in which each term is the sum of the previous two. A geometric sequence is one in which the ratio of consecutive terms is constant. We will try to create a sequence that is geometric and Fibonacci-like at the same time.

9. Consider the sequence $1, k, k^2, k^3, \ldots$. Explain why it is a geometric sequence. What is its common ratio?

10. Explain why, if $1, k, k^2, k^3, \ldots$ were a Fibonacci-like sequence, we would have $1 + k = k^2$.

11. Find a number k that satisfies the equation $1 + k = k^2$. Explain your reasoning.

In problems 9-11, you have shown that the sequence $1, \varphi, \varphi^2, \varphi^3, \ldots$ is geometric and starts out as a Fibonacci-like sequence, since its third term is the sum of the first two. It remains to show that if you add the second and third terms, you get the fourth, if you add the third and fourth, you get the fifth, and so on. More generally, we need to show that if you add the $(n + 1)^{\text{th}}$ term and the $(n + 2)^{\text{th}}$ term, you get the $(n + 3)^{\text{th}}$ term.

12. Use algebra to explain why if $1 + k = k^2$, then $k + k^2 = k^3$.

13. Multiply both sides of $1 + k = k^2$ by k^n. Use the result to show that the sequence $1, \varphi, \varphi^2, \varphi^3, \ldots$ is Fibonacci-like.

GOLDEN WINDOWS

Some architects think that rectangular windows look best if their sides are in the ratio of approximately φ.

14. Imagine that you must make "golden windows" out of square panes. Since the sides must be whole numbers, you will not be able to have an exact golden rectangle, so try to find the dimensions of a few windows having whole number sides in a ratio close to φ.

Many architects use consecutive numbers in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ... as the dimensions of windows and other rectangles. (Example: 3 by 5, or 5 by 8.)

15. Make a sequence of the ratios of consecutive Fibonacci numbers: 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, ... Are the ratios greater or less than the golden ratio? What is the trend in the long run?

16. a. Plot the points (1, 1), (1, 2), (2, 3), (3, 5), (5, 8), (8, 13), ...
b. Graph the line $y = \varphi x$.
c. Describe the position of the points in relation to the line.

17. Plot the points $(1, \varphi), (\varphi, \varphi^2), (\varphi^2, \varphi^3), \ldots$ and the line $y = \varphi x$. Compare the graph with the one in problem 16.

18. **Research** Read about the golden ratio, the golden rectangle, and the Fibonacci sequence. Write a report on what you learn.

THINKING WRITING

14.B Up and Down Stream

BOATS AND CURRENTS

The L.A. Barge Company operates boats on canals, lakes, and rivers. One of their boats, the *Huck Finn*, moves at a maximum rate of 11 mi/hr in still water. The boat regularly does a round trip on the Leumas River, going 32 miles upstream, and returning. The river flows at a rate of 2 mi/hr.

To calculate the total time for the round trip, you need to use the formula

$$\text{distance} = \text{rate} \cdot \text{time}.$$

Assuming the boat goes at its maximum rate, it goes upstream at a rate of $(11 - 2)$ mi/hr, and it goes downstream at a rate of $(11 + 2)$ mi/hr.

1. What is the total time for the round trip? Assume a one-hour stop before heading back.

2. What is the average speed
   a. with a stop?
   b. without a stop?

3. True or False? Since the boat goes upstream on the way there, and downstream on the way back, the effect of the current is cancelled, and the trip takes as long as it would on a lake. Explain.

4. How long does the upstream portion of the trip take? How about the downstream portion?

For problems 5 and 6 assume the boat moves at a rate of r miles per hour in still water.

5. What would its rate be in terms of r,
   a. going upstream if the river is moving at 2 miles per hour?
   b. going downstream if the river is moving at c miles per hour?

6. If the river is moving at 3 miles per hour,
   a. how long does the upstream portion of the trip take in terms of r?
   b. how long does the downstream portion of the trip take in terms of r?
   c. how long does the whole trip take in terms of r?

7. How fast should the boat go (still water rate), if the L.A. Barge Co. wants to conserve fuel, but needs to make the round trip (including a one-hour stop) in:
   a. 13 hours? b. 8 hours?

AIRPLANES AND WINDS

An airplane flies from Alaberg to Bergala with a headwind of 20 miles per hour and returns with a tailwind of 20 miles per hour. The plane stopped in Bergala for an hour. The whole trip took 4 hours. The towns are 500 miles apart.

8. How long did each portion of the trip take?

YOUR OWN PROBLEM

9. Create a problem involving currents, winds, or moving sidewalks that requires solving a quadratic equation. Solve your problem.

Essential Ideas

WINDOWS AND PANES

1. The A.B. Glare Window Store sells a two-pane window, especially designed so that the panes have the same dimensions as each other, and the whole window has the same proportions as each pane. If the horizontal dimension of the window is 36 inches, what is the vertical dimension, to the nearest inch? Make a sketch and show your work.

2. The A.B. Glare Window Store sells two models of two-pane windows, such that one pane is square and the other is rectangular. The rectangular pane has the same proportions as the whole window. Both models have a horizontal dimension of 36 inches. Make a sketch and show your work as you answer the following question: What are the dimensions of the rectangular pane, if its longer dimension is
   a. horizontal? b. vertical?

ALGEBRAIC FRACTIONS

3. Dwight was simplifying $\frac{x+2}{x}$. He said, "I can't get rid of the x's in the numerator and denominator." He wrote $\frac{x+2}{x} = 2$.

   Did Dwight correctly simplify $\frac{x+2}{x}$? Is his statement always, sometimes, or never true?

If possible, simplify the fractions.

4. $\frac{xy+y}{y}$
5. $\frac{3x+3y}{x^2-y^2}$
6. $\frac{3a+3b}{4a+4b}$
7. $\frac{6}{6x-6}$
8. $\frac{x^2+5x}{x^2+4x}$
9. $\frac{2x+2y}{3x+3y}$

ALWAYS, SOMETIMES, NEVER

Tell whether each expression is always, sometimes, or never true.

10. $\frac{3x+5}{3x} = 5$
11. $\frac{3x+3y}{x+y} = 6$
12. $\frac{3x+3y}{x+y} = 3$
13. $\frac{3x+y}{y} = 3x$

EQUIVALENT FRACTIONS

14. Write a fraction having a denominator of $6y$ that is equivalent to:
    a. 1/6 b. x

15. Write a fraction having a denominator of y that is equivalent to:
    a. $6x$ b. $6xy$

16. a. Write a fraction equivalent to $3/x$ having xy as a denominator.
    b. Write a fraction equivalent to $5/y$ having xy as a denominator.
    c. Add $3/x$ and $5/y$. (Hint: To add, you need a common denominator.)

17. a. Write a fraction that is equivalent to x having x as a denominator.
    b. Add $x + 1/x$. (Hint: Find a common denominator.)

Put on the same denominator.

18. $x^2 + \frac{b}{a}x + \frac{c}{a}$
19. $-\frac{b^2}{4a^2} + \frac{c}{a}$

DIVISION BY ZERO?

On a test Joel solved the quadratic equation $6x^2 = 12x$ using this method:

Divide both sides by x:	$\frac{6x^2}{x} = \frac{12x}{x}$
Simplify fractions:	$6x = 12$
Divide both side by 6:	$\frac{6x}{6} = \frac{12}{6}$
The answer is	$x = 2$.

Joel's teacher, Mr. Letter, wrote this on his paper:

> There are two solutions to this equation. You missed one of them because you divided by 0.

Joel was puzzled. "I divided by x, and then by 6" he thought. "I never divided by 0."

20. Can you explain what Mr. Letter meant? Can you solve the equation correctly?

MYSTERY PARABOLAS

Make a rough sketch showing two parabolas having the features described. Some of your parabolas should be frowns and others smiles; some should be more open, some less. Label each parabola with:

a. its equation;
b. its axis of symmetry;
c. its x-intercepts (exact values);
d. its vertex.

21. The parabola has x-intercepts at 2 and -4.

22. The parabola has vertex (3, -5).

23. The parabola has an x-intercept at $\sqrt{5}$ and is symmetric with respect to the y-axis.

24. The parabola has an x-intercept at $1 - \sqrt{6}$ and has the line $x = 1$ as its axis of symmetry.

25. The parabola has an axis of symmetry at $x = 5$ and y-intercept 3.

PARABOLA FEATURES

26. Give the vertex, x-, and y-intercepts of:
a. $y = 2(x + 3)^2 - 9$
b. $y = 4(x - 5)(x + 1)$
c. $y = 6x^2 - 7x - 8$

27. How many x-intercepts?
a. $y = -2(x + 3)^2 - 9$
b. $y = -4(x - 2)$
c. $y = 6x^2 + 7x + 8$

FROM FRACTIONS TO QUADRATICS

Rewrite each equation as an equivalent quadratic equation. Then solve the equation. Show your work.

28. $w + 9 = \frac{10}{w}$ **29.** $L + 3 = 2 + \frac{6}{L}$

30. $L - 4 = \frac{32}{L}$ **31.** $\frac{1}{x} = x - 1$

Solve these equations. They have zero, one, or two solutions.

32. $\frac{4}{x} + x = -4$ **33.** $\frac{1}{x} + \frac{2}{x} = \frac{3}{x}$

34. $1 = \frac{1}{x} + \frac{1}{x^2}$

WRITE AN EQUATION

35. Write a quadratic equation that has the following solutions:
a. 4 and -2
b. $\sqrt{5}$ and $-\sqrt{5}$
c. $1 + \sqrt{5}$ and $1 - \sqrt{5}$

36. Write a quadratic equation that has the solution -6.

37. Write a quadratic equation that has no real number solutions.