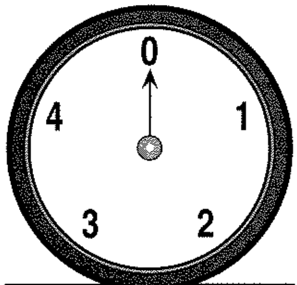
6.C Group Theory

MOD CLOCKS



Mod 5 Clock

The figure shows a mod clock, which is a special function machine. For any positive whole number input, it will output a number between 0 and 4. For example:

Input	Output	Input	Output
1	1	5	0
9	4	12	2
13	3	17	2
25	0	26	1
77	2	100	0

1. What would be the output of the mod clock for the following inputs? Explain.
a. 1998 b. 1899 c. 9981

Definition: $a \oplus b$ is the output from the mod clock for the input $a + b$. $a \otimes b$ is the output for the input ab.

Example: $3 \oplus 2 = 0$, and $3 \otimes 2 = 1$

2. Make a table for each of $\oplus$ and $\otimes$.

3. Generalization The clock above is a mod 5 clock. Find ways to predict the output of mod 10, mod 2, mod 9, and mod 3 clocks.

GROUPS

Definition: A *group* is a set of elements, together with an operation that satisfies the following rules.

- *closure:* using the operation on two elements of the group yields an element of the group.
- *associative law:* $(ab)c = a(bc)$.
- *identity element:* one of the elements, e, is such that $ae = ea = a$, for any element a in the group.
- *inverse element:* every element a has an inverse a' such that $a\,a' = a'a = e$

Some groups are *commutative* ($ab = ba$) and some are not.

For 4-7 assume the associative law holds.

4. a. Show that the set $\{0, 1, 2, 3, 4\}$ together with the operation $\oplus$ is a group.
b. Show that $\{0, 1, 2, 3, 4\}$ with $\otimes$ is not a group.
c. Show that $\{1, 2, 3, 4\}$ with $\otimes$ is a group.

5. Is the set of the integers a group with the following operations?
a. addition b. multiplication

6. Show that the set of rational numbers (positive and negative fractions and zero) together with multiplication is not a group. By removing one element, it can be made into a group. Which element? Explain.

7. Think about a mod 4 clock, with the numbers $\{0, 1, 2, 3\}$. Is it a group for $\oplus$? For $\otimes$? Can it be made into one by removing an element?

8. Report Give examples of groups. For each, give the set and operation. Explain how they satisfy the rules. Include finite, infinite, commutative, and noncommutative groups.