LESSON 11.2

Decimals and Fractions

WRITING FRACTIONS AS DECIMALS

1. How do you convert a fraction to a decimal number? Give examples.

When converting fractions to decimals, sometimes you get a *terminating* decimal like 3.4125, and sometimes you get a *repeating* decimal, like 7.8191919.... This last number is often written $7.8\overline{19}$.

Problems 2 and 3 are easier if you work with lowest-term fractions.

2. **Exploration** For what fractions do you get a repeating decimal? Does it depend on the numerator or the denominator? (Hint: Pay attention to the prime factorization of the numerator and the denominator.)

3. **Exploration** For repeating decimals, is there a pattern to the number of digits in the repeating part? What is the longest possible repeating string for a given denominator? (Hint: Use long division rather than a calculator to explore this.)

4. Explain why the decimals obtained as a result of a division *must* repeat or terminate.

5. Explain why some calculators give a decimal that does not seem to repeat for 2/3: 0.6666666667.

WRITING DECIMALS AS FRACTIONS

Example: 3.4125 can be converted to a fraction by multiplying it by 10^4, which gets rid of the decimal, and then dividing by 10^4, which gets us back to the original number.

$$\frac{34{,}125}{10{,}000}$$

6. Convert these decimals to fractions.
   a. 6.0 b. 3.2
   c. 0.015 d. 3.41

The case of repeating decimals is more difficult. Take $7.8\overline{19}$. Clearly, it is greater than 7.81 and less than 7.82. So it is between 781/100 and 782/100.

To find a single fraction it is equal to, we can rewrite it as:

$$\begin{aligned}&7.80\overline{19}\\&= 7.8 + 0.0\overline{19}\\&= 7.8 + 0.019 + 0.00019 + 0.0000019 + \ldots\end{aligned}$$

Observe that:

$$0.00019 = 0.019(0.01)$$
$$0.0000019 = 0.019(0.01)^2$$

7. Write the next term in the sum as a decimal, and as a product of 0.019 and a power of 0.01.

As you see, $7.8\overline{19}$ is the sum of 7.8 and a geometric sequence with first term 0.019 and common ratio 0.01. The sum of the first three terms of the geometric sequence can be written:

$$S = 0.019 + 0.019(0.01) + 0.019(0.01)^2$$

Multiply both sides by 0.01:

$$S(0.01) = 0.019(0.01) + 0.019(0.01)^2 + 0.019(0.01)^3$$

Subtract:

$$S(1 - 0.01) = 0.019 - 0.019(0.01)^3$$

Solve:

$$S = \frac{0.019 - 0.019(0.01)^3}{0.99}$$

Multiplying numerator and denominator by 1000:

$$S = \frac{19 - 19(0.01)^3}{990}$$

$$7.8\overline{19} = 7.8 + S$$
$$= 7.8 + \frac{19 - 19(0.01)^3}{990}$$
$$= \frac{7.8(990) + 19 - 19(0.01)^3}{990}$$

So

$$= \frac{7741 - 19(0.01)^3}{990}$$
$$= \frac{7741 - 0.000019}{990}$$

The sum is very close to 7741/990.

8. Use the multiply-subtract-solve technique to add:

a. the first 4 terms;

b. the first 5 terms.

9. The numerator differs from 7741 by $19(0.01)^n$ if we add up the first n terms. Explain.

If we use large values for n, we find that the sum can get as close to 7741/990 as we want. (Even with fairly small values of n, the sum of the first n terms differs from 7741/990 by a *very* small number.) Mathematicians say that the whole infinite sum *converges* to 7741/990, and they agree that we can write an equality:

$$7.8\overline{19} = 7741/990.$$

10. Check that this equality is correct, by converting the fraction back to a decimal.

A quick way to find the fraction is to use the multiply-subtract-solve technique on the decimal itself:

$$R = 7.8191919...$$
$$0.01\,R = 0.0781919...$$

Subtract:

$$R - 0.01R = 7.8191919... - 0.0781919...$$
$$(1 - 0.01)R = 7.819 - 0.078$$

(Notice that the infinite sequence of 19s disappeared.)

$$0.99R = 7.741$$
$$R = \frac{7.741}{0.99} = \frac{7741}{990}$$

11. Convert to a fraction.

a. $0.\overline{65}$ b. $4.\overline{321}$

RATIONAL NUMBERS

Definition: A *rational number* is a number that can be written as a fraction having an integer numerator and denominator.

Examples: 7, 0.5, and -0.66666... are rational numbers, because they can be written as 7/1, 1/2, and -2/3.

Show that the following numbers are rational.

12. a. 0.3

b. 0.3333...

13. a. 0.142857

b. $0.\overline{142857}$

14. a. 0.0909090...

b. 0.9090909...

15. a. 0.1111111...

b. 0.2222222...

16. Mathematicians believe that 0.99999... = 1. Explain why.