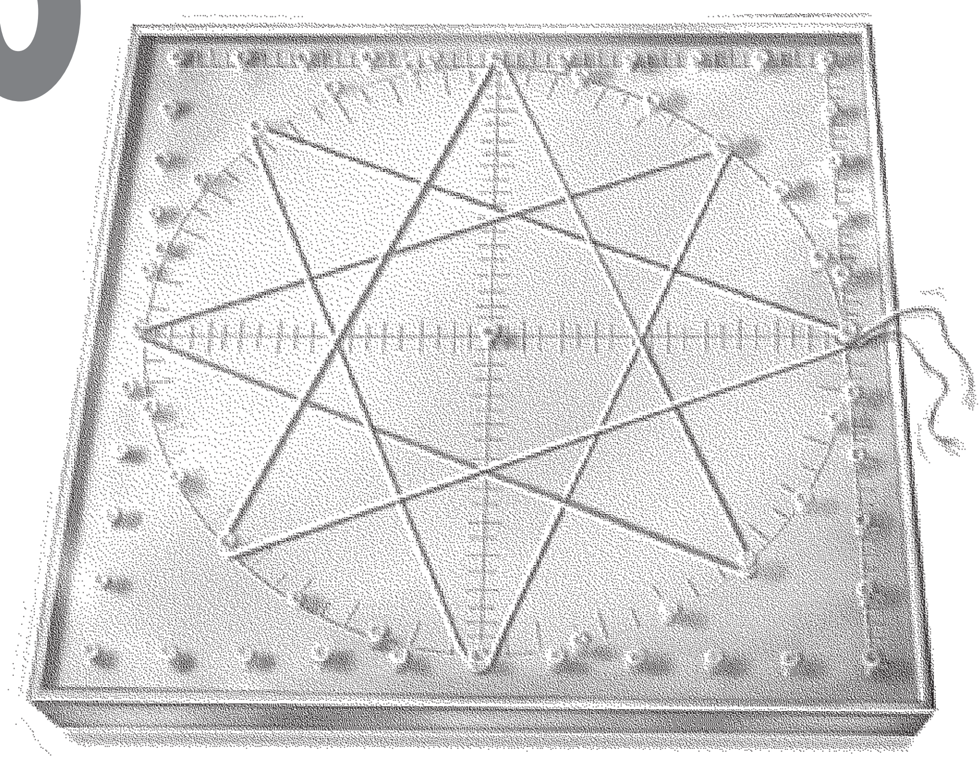
3 Polygons



In the first two labs, the students construct triangles, given their sides and given their angles. In addition to the overt topics (the triangle inequality and the sum of the angles in a triangle), these labs allow you to introduce your chosen approach to construction (see the teacher notes to Lab 3.1). Moreover, they provide a first preview of two big ideas: congruence and similarity.

Labs 3.3, 3.5, and 3.6 ground students' understanding of angles by having them "walk around" the perimeters of various polygons. This complements the work from Section 1, and it also lays the groundwork for discovering the formula for the sum of the angles of a polygon. Turtle graphics, an approach to geometry that was initiated in the context of the computer language Logo, was the original inspiration for polygon walks. See the teacher notes pertaining to Lab 3.5 (Walking Regular Polygons) and Lab 3.6 (Walking Nonconvex Polygons).

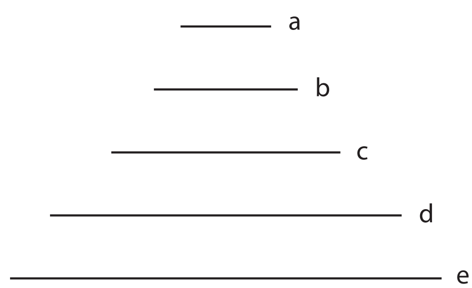
The section ends with additional lessons on the sum of the angles of a polygon, which provide interesting algebraic connections.

See page 176 for teacher notes to this section.

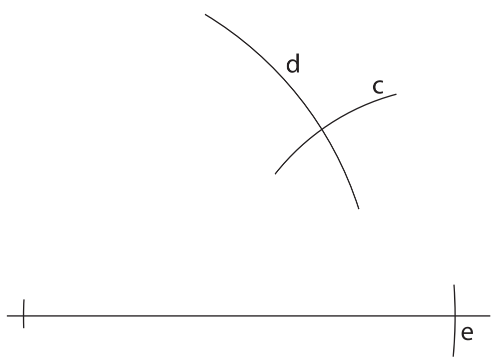
LAB 3.1
Triangles from Sides

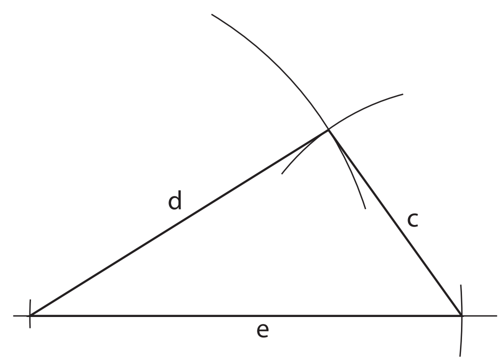
Name(s) ______________________

■ **Equipment:** Compass, straightedge, unlined paper



1. Here is a technique for constructing a triangle with sides equal to *c, d,* and *e* above. Practice this technique on a separate paper.
   - **a.** Copy side *e*. The endpoints give you two vertices of the triangle.
   - **b.** Make an arc centered at one end, with radius *d*.
   - **c.** Make an arc centered at the other end, with radius *c*.
   - **d.** Where the two arcs meet is the third vertex. Connect it to the other two. You are done.





2. Using the same technique, construct at least three triangles with sides equal to *a, b, c, d,* or *e*. Try to make triangles different from those of your neighbors.
3. Try to construct a triangle with sides equal to *a, b,* and *e*. In the space below, explain why it is impossible.

4. Working with other students, make a complete list of possible triangles using sides *a, b, c, d,* and *e*. Include equilateral, isosceles, and scalene triangles. Also make a list of impossible triangles.

Possible (example: *c,d,e*) **Impossible** (example: *a,b,e*)

Discussion

A. Would it be possible to construct a triangle with side lengths 2, 4, and 8.5? Explain.

B. Would it be possible to construct a triangle with side lengths 2, 4, and 1.5? Explain.

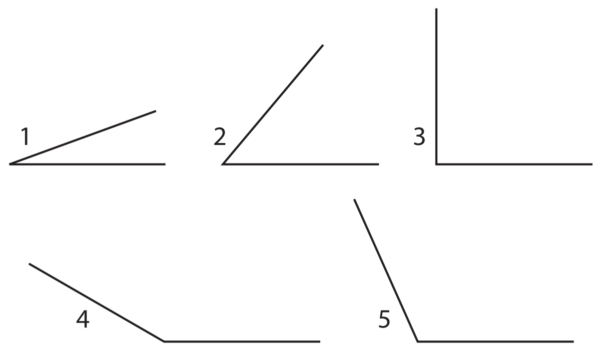
C. If two of the sides of a triangle have lengths 2 and 4, what do you know about the third side?

D. State a generalization about the lengths of the three sides of a triangle.

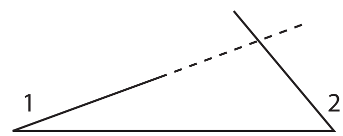
LAB 3.2
Triangles from Angles

Name(s) ______________________

■ **Equipment:** Compass, straightedge, unlined paper



1. Here is a technique for constructing a triangle with angles equal to ∠1 and ∠2. Follow these steps and practice this technique on a separate sheet of paper.



a. Copy ∠1. That gives you one vertex.

b. Choose any point on a side of ∠1. That will be the second vertex.

c. Copy ∠2 at the second vertex. (Note that in the figure at right above, it was necessary to flip ∠2 over.)

d. Extend the other sides of ∠1 and ∠2. They meet at the third vertex. You are done.

2. Using the same technique, construct at least three triangles by copying two angles from among ∠1, ∠2, ∠3, ∠4, and ∠5. Try to make triangles different from those of your neighbors.

3. Would it be possible to construct a triangle with angles equal to ∠4 and ∠5? Explain in the space below.

4. Make a list of possible triangles using two angles chosen from among $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, and $\angle 5$ (including obtuse, right, and acute triangles). It's okay to use two copies of the same angle. Also make a list of impossible triangles.

Possible (example: $\angle 1$, $\angle 2$) **Impossible**

Discussion

A. Given three angles, is it usually possible to construct a triangle with those angles? What must be true of the three angles?

B. What must be true of two angles to make it possible to construct a triangle with those angles?

C. Given two angles, $\angle 1$ and $\angle 2$, describe two different ways to construct the angle $180° - (\angle 1 + \angle 2)$.

LAB 3.3

Name(s) ______________________

Walking Convex Polygons

■ **Equipment:** Pattern blocks, template, unlined paper

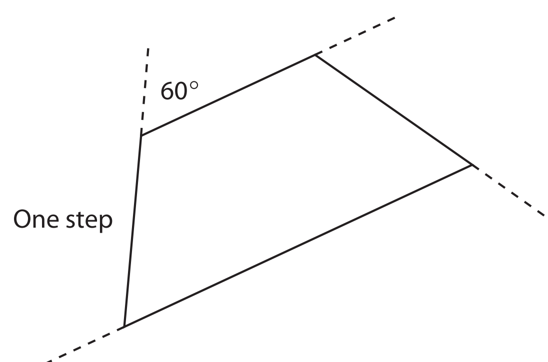
Definitions:

A *polygon* is a closed figure with straight sides.
An *n-gon* is a polygon with *n* sides.
A *convex* polygon is one where no angle is greater than 180°.

1. Is there such a thing as a nonconvex triangle? If so, sketch one in the space at right; if not, explain why not.
2. Is there such a thing as a nonconvex quadrilateral? Sketch one, or explain why there's no such thing.
3. For each number of sides from 3 to 12, make a *convex* pattern block polygon. Use your template to record your solutions on a separate sheet.

In this activity, you will give each other instructions for walking polygons. As an example, here are instructions for walking the pattern block trapezoid. We will use a scale of one step for one inch:

Walk forward one step from the starting point;
Turn right 60°;
Walk forward one step;
Turn right 60°;
Walk forward one step;
Turn right 120°;
Walk forward two steps;
Turn right 120°.



The final turn is not necessary, since the whole perimeter has been walked by then. However, it is traditional to include it, because it brings the walker back exactly to the starting position.

4. Mark the remaining steps and angles on the figure.

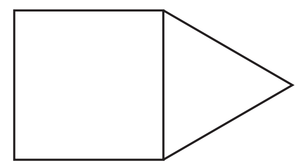
Note: The turn angle is also called the *exterior angle*. It is not usually the same as the interior angle, which is usually called simply the *angle*.

5. Write another set of instructions for walking the trapezoid, starting at another vertex.

6. On a separate paper, write instructions for walking each of the other pattern blocks.

7. Take turns with a partner, following each other's instructions for walking different blocks.

8. Make a two-block *convex* pattern block polygon, such as the one in the figure at right, and write instructions below for walking it.



Discussion

A. Write alternate instructions for walking the trapezoid with one or more of these additional constraints:
- Walking backward instead of forward
- Making left turns instead of right turns
- Starting in the middle of the long side

B. What is the relationship between the turn angle and the interior angle? When are they equal?

C. How can you tell whether a polygon walk is clockwise or counterclockwise by just reading the walk's instructions?

LAB 3.4

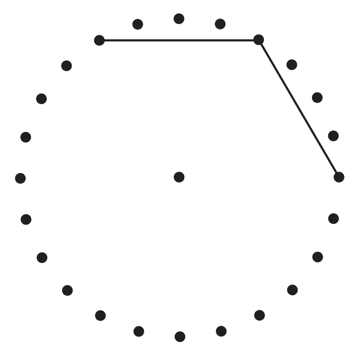
Name(s) ____________________________

Regular Polygons and Stars

- **Equipment:** Circle geoboard, string, Circle Geoboard Record Sheet

Definition: A *regular polygon* is a polygon in which all the angles are equal to each other and all the sides are equal to each other.

1. A regular triangle is called a(n) ______________ triangle. A regular quadrilateral is called a(n) ___________.
2. Tie a string at the 0° peg on the geoboard. Then make a figure by going around every fourth peg. The beginning of this process is shown in the figure at right.



   Keep going until you are back to the starting peg.

   a. What sort of figure did you get? ___________
   Sketch it on a circle geoboard record sheet.

   b. What is the measure of each angle of your figure?
   ____

3. Repeat Problem 2, using every ninth peg.

Every p-th peg	Star or polygon?	Number of sides	Angle measure
1			
2			
3			
4	polygon	6	120°
5			
6			
7			
8			
9	star		
10			
11			
12			

Every p-th peg	Star or polygon?	Number of sides	Angle measure
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			

4. Repeat Problem 2 by using every peg, every other peg, every third peg, and so on, all the way to every twenty-fourth peg. Look for patterns. You won't have enough string to make some of the stars on the geoboard, but you should still draw them on your record sheet. Record your entries in the table on the previous page, and record below any patterns you discover.

Discussion

A. Which of the pattern blocks are regular polygons?

B. Explain why the blue pattern block is not a regular polygon.

C. Make a pattern block polygon that has all angles equal but is not regular.

D. What patterns did you discover while filling out the table?

E. How can you predict whether using every p-th peg will give a star or a polygon?

F. Describe the star that you think would require the most string. Describe the star (not polygon) that would require the least string.

G. Discuss the situation when $p = 12$ or $p = 24$.

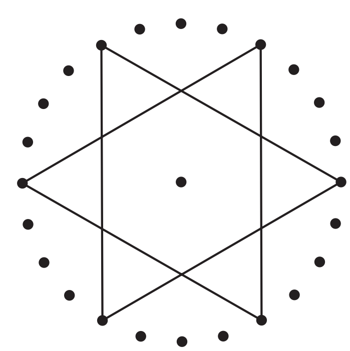
H. Explain why there are two values of p that yield each design.

I. How can you predict the angle from the value of p? Give a method or a formula.

J. Repeat this exploration on an imaginary 10- or 20-peg circle geoboard.

K. For a circle geoboard with n pegs, which p-th pegs will produce a star, and which will produce a polygon?

L. Investigate stars that use two strings or that require you to pick up your pencil, like the one shown at right.



LAB 3.5

Name(s) ____________________

Walking Regular Polygons

Here are three descriptions of how to walk a square.

Jenny: Turn 90° left and take a step four times.
Maya: (Take step, turn right 90°) · 4
Pat: (1) Step forward; (2) turn right 90°; (3) step forward; (4) turn right 90°; (5) step forward; (6) turn right 90°; (7) step forward.

1. What is the *total turning* in degrees for each set of directions? (They are not all the same!) Jenny: __________ Maya: ____________ Pat: ___________

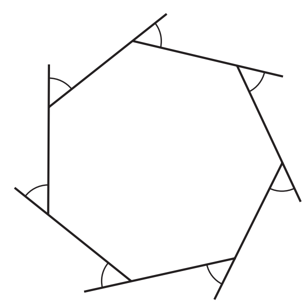
2. Use Maya's style to write instructions for an equilateral triangle walk.

3. What is the total turning for the triangle? _________

4. a. When we say that the sum of the angles in a triangle is 180°, what angles are we adding?

b. When we calculate the total turning, what angles are we adding?

5. These problems are about a regular 7-gon.



a. What is the measure of each exterior angle? Explain.

b. What is the measure of each interior angle? Explain.

6. Fill out this table for regular polygons.

Number of sides	Each angle	Angles' sum	Turn angle	Total turning
3	60°	180°		
4	90°	360°		
5				
6				
7				
8				
9				
10				
11				
12				
100				
n				

Discussion

A. Compare the three sets of instructions for walking a square.

- **a.** Do they all work?
- **b.** How are they the same?
- **c.** How are they different?
- **d.** Which do you think is the clearest?
- **e**. How would you improve each of them?

B. Which column of the table in Problem 6 is easiest to find? How can this help you find the numbers in the other columns?

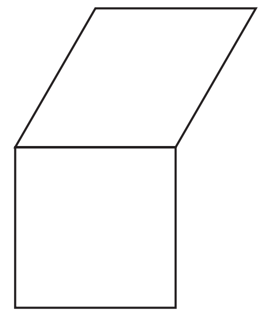
C. Compare the formulas you found with those of other students.

D. Explore the angles for regular stars: the sum of interior angles; the sum of turn angles. Can you find patterns? Formulas?

LAB 3.6
Walking Nonconvex Polygons

Name(s) ______________________

■ **Equipment:** Pattern blocks



1. Use two pattern blocks to create a nonconvex polygon such as the one in the figure at right.

a. Write instructions for walking it, assuming that the walker begins and ends facing the same direction.

b. Follow a neighbor's instructions, and have him or her follow yours.

2. Since the walker ends up facing the way he or she started, what should the *total turning* be?

a. Calculate the total turning to check whether your prediction worked. Show your calculation below.

b. Check a neighbor's total turning.

The direction you are facing is called your *heading*. North is heading 0°. Other headings are measured in degrees clockwise from North. East is 90°.

3. What is your heading if you are facing the following directions?

a. South __________

b. West __________

c. Southwest __________

d. NNW __________

4. Simone and Joy are standing back to back. Find Simone's heading if Joy's is:

a. 12° __________

b. 123° __________

c. h° (careful!) __________

5. How would you interpret a heading greater than 360°?

6. How would you interpret a heading less than 0°?

7. Assume you start walking your nonconvex polygon facing North. What is your heading while walking each side?

8. Does turning left add to or subtract from your heading? What about turning right?

9. In view of Problem 8, how should you deal with turn angles to get results that are consistent with changes in heading? With this interpretation, what is the total turning for a polygon, whether convex or not? Explain.

Discussion

A. In Problem 2, the total turning should be 360°. How could we calculate total turning so that this works? (**Hint:** The solution involves using positive and negative numbers.)

B. What does turning 360° do to your heading? What about turning 350°? How is this related to the discussion about positive and negative turns?

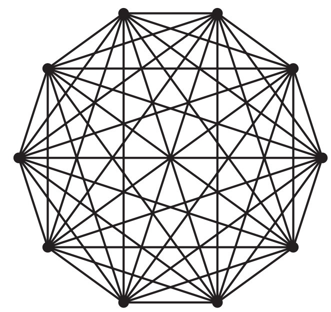
C. What does it mean to turn right –90°? Can the same turn be accomplished with a positive left turn? A positive right turn?

D. If we want total turning to be 360° and want our turns to be consistent with changes in heading, should we walk polygons in a clockwise or counterclockwise direction?

LAB 3.7
Diagonals

Name(s) ____________________

Definition: A *diagonal* is a line segment connecting two *nonconsecutive* vertices of a polygon.



The figure at right is a regular decagon, with all its diagonals. How many diagonals are there? It is not easy to tell. In this activity, you will look for a pattern in the number of diagonals in a polygon. You will also use diagonals to think about the sum of the angles in a polygon.

1. How many diagonals does a triangle have? __________
2. How many diagonals does a quadrilateral have? __________
3. Fill out this table, stating the total number of diagonals.

Sides	Diagonals
3	
4	
5	
6	

Sides	Diagonals
7	
8	
9	
100	

4. Find a formula for the total number of diagonals in an n-gon. ____________

In the remainder of the lab, we will limit our discussion to convex polygons, so all diagonals will be on the inside of the polygon.

5. If you draw all the diagonals *out of one vertex* in a convex n-gon, you have divided it into triangles—how many? (Experiment with 3-, 4-, 5-gons, and so on, and generalize.) __________
6. Use the answer to the previous problem to find a formula for the sum of the angles in a convex n-gon. __________
7. Explain why this reasoning does not work for a nonconvex n-gon.

Discussion

A. What patterns do you see in the table in Problem 3?

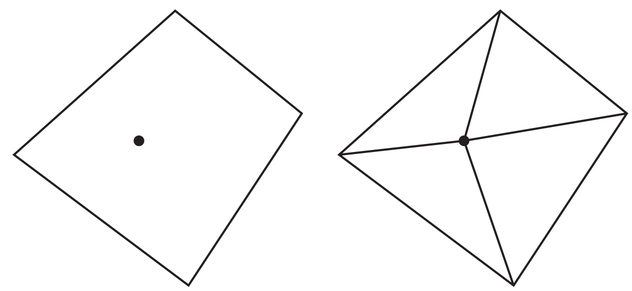
B. Experiment with convex and nonconvex polygons and their diagonals. Write a new definition of *convex polygon* that uses the word *diagonals*.

LAB 3.8

Name(s) ____________________

Sum of the Angles in a Polygon

1. The sum of the interior angles of *any* quadrilateral is always the same. Here is an explanation of why that's true, based on dividing the quadrilateral into triangles.



a. Draw any random quadrilateral (not a square).

b. Put a point inside it, and connect it to each vertex.

c. What is the sum of all the angles in the four triangles? __________

d. How much of that sum is around the vertex at the center? __________

e. How much of that sum is the sum of the angles in the quadrilateral? __________

2. a. Use the same logic to find the sum of the angles in a 5-, 6-, 8-, and 12-gon.

5-gon __________

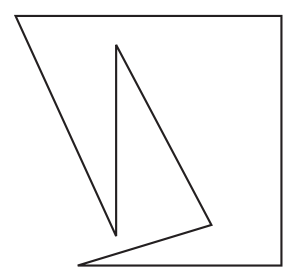
6-gon __________

8-gon __________

12-gon __________

b. Use the same logic to find a formula for the sum of the interior angles in an *n*-gon.

3. Use the figure at right, or one you create, to explain why the logic of Problems 1 and 2 does not work for all polygons.



4. Find a way to divide the polygon in Problem 3 into triangles in order to find the sum of its angles. Record this sum. ______________

5. Is your answer to Problem 4 consistent with the formula you found in Problem 2?

Discussion

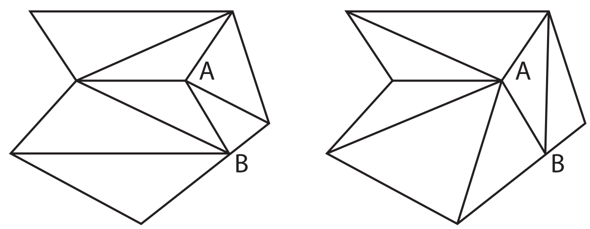
A. Compare the method in Problem 1 for finding the sum of the angles in a polygon with the one in Lab 3.7. Do they yield the same formula? Do they use the same numbers of triangles?

LAB 3.9

Name(s) ______________________________

Triangulating Polygons

Definition: To *triangulate* a polygon means to divide it into triangles. Triangle vertices can be vertices of the original polygon, or they can be new points, either on the inside of the polygon or on one of the sides.



For example, the hexagon above has been divided into seven triangles in two different ways, both of them using one inside vertex (*A*) and one side vertex (*B*).

To triangulate a polygon, decide how many additional vertices you will use and where you will put them. Then connect vertices in any way you want, as long as the segments you draw never cross each other, do not pass through a vertex, and the polygon ends up divided into triangles.

Experiment with triangulating polygons. For each triangulation, enter the information in a table like the one below. Look for patterns. On a separate paper, write several paragraphs describing your findings. These questions may help you organize your research:

- How many triangles are added when you add 1 to the number of sides and keep other things constant?
- How many triangles are added when you add 1 to the number of inside vertices?
- How many triangles are added when you add 1 to the number of side vertices?
- Can you find a formula relating the number of original vertices (n), the number of inside vertices (i), the number of side vertices (s), and the number of triangles (t)?

The data for the triangulations above have been entered for you.

Original vertices	Inside vertices	Side vertices	Number of triangles
6	1	1	7

Discussion

A. In the example given, is it possible to get a different number of triangles by moving A to another location inside the hexagon and/or moving B to another location on one of the sides?

B. How would you define a minimal triangulation for a polygon? How might you draw it?

C. If the vertices are distributed differently, how does that affect the total number of triangles? For example, if you have a total of eight vertices, as in the example on the previous page, but with four original vertices, two inside vertices, and two side vertices, do you still have seven triangles? If not, how would you distribute the vertices to get the smallest number of triangles possible? What is that number? How would you distribute the vertices to get the greatest number of triangles possible? What is that number?

D. How is triangulation related to the sum of the angles in a polygon?