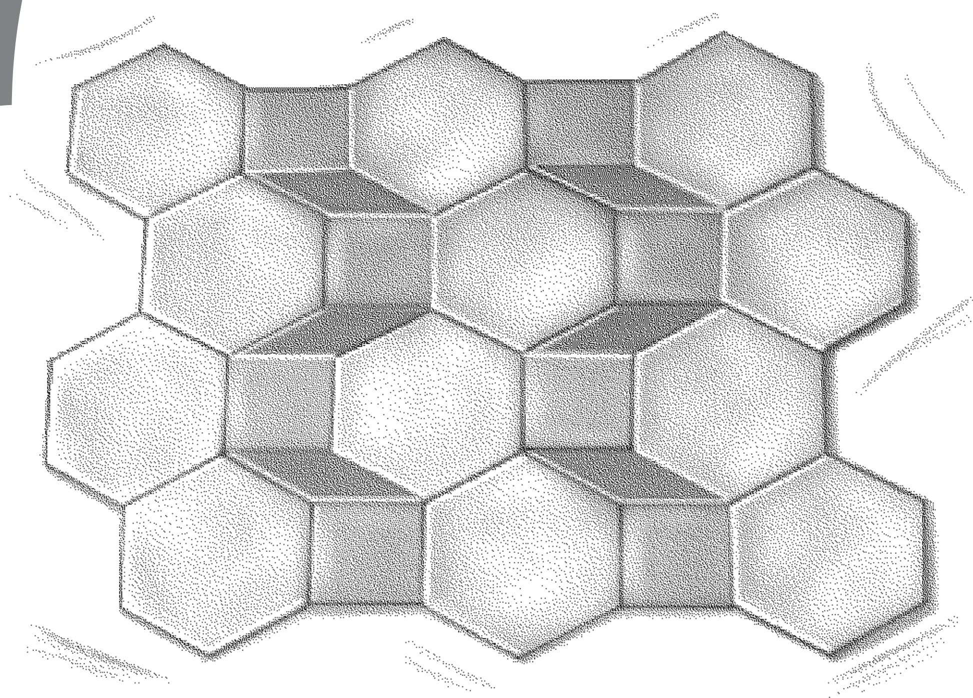
7 Tiling



In this section, we address one question from several points of view: What are some ways to tile a plane? *To tile a plane* means to cover an infinite flat surface with an unlimited supply of a finite number of shapes, with no gaps or overlaps. Some types of tilings are also called *tessellations*. To keep jargon to a minimum, we will use *tiling* rather than *tessellation*.

Some of the resulting investigations are directly connected with the traditional high school geometry curriculum and reinforce standard topics, particularly Lab 7.3 (Tiling with Triangles and Quadrilaterals) and Lab 7.4 (Tiling with Regular Polygons).

Every lab in this section lends itself to the creation of a bulletin board display of student work, featuring the best-looking tilings and some explanations of the underlying mathematics.

You may not have enough time to do all four labs on this topic, or you may suspect that your students would find them to be too much on the same topic. If so, you may use Lab 7.1 (Tiling with Polyominoes) to introduce the concepts and distribute the remaining labs among your students, or groups of students, as independent projects, which can be combined in a final class exhibit. Many students are fascinated with M. C. Escher's graphic work, which involves tiling with figures such as lizards, birds, or angels. A report on Escher could be one of the independent projects.

Note that tilings provide interesting infinite figures for students to analyze from the point of view of symmetry. To support and encourage that inquiry, the same symmetry questions are asked about each tiling in each discussion section. That kind of analysis can be a worthwhile complement to the projects suggested above.

Although tiling is as ancient as tiles, it is a fairly recent topic in mathematics. Except for Kepler, all of the research on it has occurred since the nineteenth century, and new results continue to appear today. The definitive reference on the mathematics of tiling is *Tilings and Patterns* by Banko Grunbaum and G. C. Shephard (Freeman, N.Y., 1987), which features many beautiful figures.

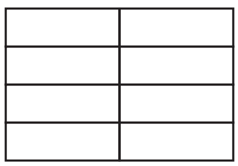
See page 203 for teacher notes to this section.

LAB 7.1
Tiling with Polyominoes

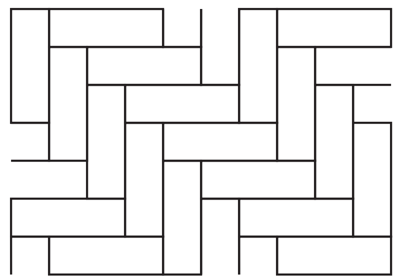
Name(s) ______________________

■ **Equipment:** Grid paper

Imagine a flat surface, extending forever in all directions. Such a surface is called a *plane*. If you had an unlimited supply of rectangular polyomino tiles, you could cover as large an area as you wanted. For example, you could lay out your tiles like this:

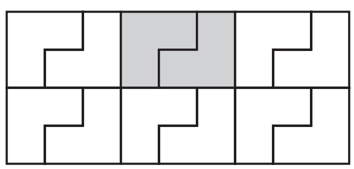


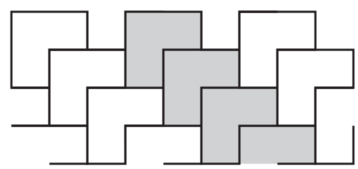
. . . or like this:



1. Find interesting ways to tile a plane with rectangular polyominoes. Record them on your grid paper.

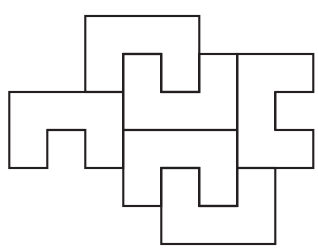
Below are two bent tromino tilings:





The first one is made of two-tromino rectangles; the second is made of diagonal strips. By extending the patterns, you can tile a plane in all directions.

2. Can you extend this U pentomino tiling in all directions? Experiment and explain what you discover.



3. Find ways to tile a plane with each of the following tetrominoes.
   a. n
   b. l
   c. t
4. Find ways to tile a plane with each of the twelve pentominoes.

Discussion

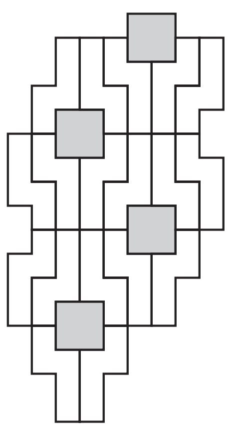
A. Would it be possible to find tilings for rectangles that are not polyominoes? (In other words, their dimensions are not whole numbers.)

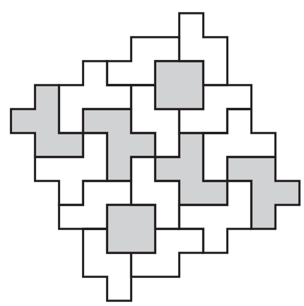
B. How can you be sure a given tiling will work on an infinite plane? Try to establish general rules for answering this question, and summarize them in a paragraph.

C. In Problems 3 and 4, you can add the constraint that the polyominoes cannot be turned over, just rotated and moved over. Is it possible to tile a plane under these conditions?

D. Polyomino tilings may involve the polyomino in one or more positions. For example, the first figure in this lab shows a polyomino that is always in the same position. The second shows a polyomino in two different positions. Can you set additional challenges for each polyomino by looking for tilings where it appears in one, two, three, or more different positions?

E. The figures below show two- and three-polyomino tilings. Create your own.





F. Analyze the symmetries of the tilings that you found. Do they have lines of symmetry? Are all of those parallel to each other? If not, what angle do they make with each other? Do the tilings have centers of rotation symmetry? How are those arranged? Do all the centers have the same n-fold symmetry?

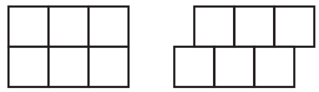
LAB 7.2

Name(s) ______________________

Tiling with Pattern Blocks

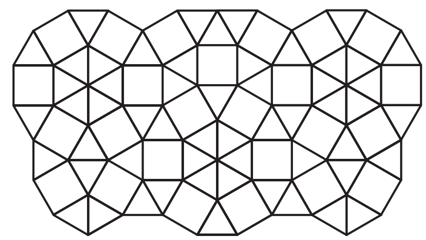
■ **Equipment:** Pattern blocks, unlined paper, template

It is easy to tile a plane with pattern blocks. Below are two possible orange tilings:



In the first one, all the vertices are o90-o90-o90-o90. In the second, all the vertices are o90-o90-o180. (The o stands for *orange;* the numbers represent the measures in degrees of the angles surrounding the vertex.)

1. Find at least two one-color pattern block tilings. Use your template to draw them on unlined paper. For each one, describe the vertices as in the examples above.
2. In some pattern block tilings, there are different types of vertices. For example, in the one at right, there are two: g60-g60-g60-g60-g60-g60 and g60-g60-o90-g60-o90. Mark the two types of vertices using different-colored dots on the figure.



3. Find and draw a pattern block tiling where all the vertices are o90-b120-o90-b60.
4. Find and draw at least two other two-color tilings. For each one, describe the vertices as above.
5. Find and draw one pattern block tiling with the following types of vertices.
   - **a.** o90-t150-b120
   - **b.** o90-b60-t30-o90-b60-t30
6. Find another three-color pattern block tiling. Describe its vertices.
7. Explain why there could not be a vertex of type r60-b120-r120 in a tiling of a plane.

Discussion

A. What must be true of the numbers in the description of a vertex? How can that help us discover new pattern block tilings?

B. Find a way to make the vertex notation more concise. Do you think that a more concise notation would be an improvement?

C. Analyze the symmetries of the tilings that you found. Do they have lines of symmetry? Are all of those parallel to each other? If not, what angle do they make with each other? Do the tilings have centers of rotation symmetry? How are those arranged? Do all the centers have the same *n*-fold symmetry?

LAB 7.3

Name(s) ____________________

Tiling with Triangles and Quadrilaterals

- **Equipment:** Template, unlined paper

Eight types of triangles

Equilateral (EQ)	Acute isosceles (AI)
Right isosceles (RI)	Obtuse isosceles (OI)
Acute scalene (AS)	Right scalene (RS)
Half-equilateral (HE)	Obtuse scalene (OS)

Eight types of quadrilaterals

Square (SQ)	Rhombus (RH)
Rectangle (RE)	Parallelogram (PA)
Kite (KI)	Isosceles trapezoid (IT)
General trapezoid (GT)	General quadrilateral (GQ)

1. Using your template on unlined paper, work with your classmates to find a way to tile a plane with each of the figures listed above.
2. Can you find a triangle that cannot tile a plane? If yes, explain how you know it will not tile. If you think any triangle can tile, explain how to do it, basing your explanation on a scalene triangle.
3. Tile a plane with a nonconvex quadrilateral.
4. Can you find a quadrilateral that cannot tile a plane? If so, explain how you know it will not. If you think any quadrilateral can tile a plane, explain how to do it, basing your explanation on a general quadrilateral.

Discussion

A. In some cases, there may be different tilings based on the same tile. Find variations on some of the sixteen tilings you found in Problem 1.

B. A parallelogram tiling can be turned into a scalene triangle tiling by drawing the diagonals of the parallelograms. In the other direction, some scalene triangle tilings can be turned into parallelogram tilings by combining pairs of triangles into parallelograms. Find other relationships of this type among tilings.

C. Analyze the symmetries of the tilings that you found. Do they have lines of symmetry? Are all of those parallel to each other? If not, what angle do they make with each other? Do the tilings have centers of rotation symmetry? How are those arranged? Do the centers all have the same *n*-fold symmetry?

LAB 7.4

Name(s) ____________________

Tiling with Regular Polygons

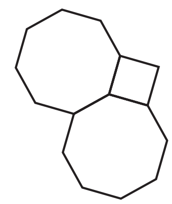
■ **Equipment:** Template, pattern blocks, unlined paper

1. Which regular polygons can be used to tile a plane? Draw the tilings on unlined paper using your template. (Multiple solutions are possible in some cases.)

2. Explain why other regular polygons would not work.

3. Find and draw a tiling that uses two different regular polygons.

In order to tile a plane with polygons, each vertex must be surrounded by angles that add up to 360°. This observation leads to a method for finding ways to tile by seeing what regular polygons can be used to surround a point. For example, two octagons and one square do work.



4. Find as many ways as possible to place regular polygons from your template around a point with one vertex of each polygon touching the point. List the ways you find below.

5. Some of the answers to Problem 4 lead to possible tilings of a plane. Find and draw some of them.

6. Some of the answers to Problem 4 do not lead to a possible tiling of a plane. Choose one and explain why it does not work.

Name(s) ____________________________

Discussion

A. Some regular polygons, or combinations of regular polygons, can be put together to make smooth strips (infinite strips with straight edges). Those can then be combined in any number of ways to make tilings of a plane. Find such strips.

B. There are five other ways to use regular polygons to surround a point. They involve a 15-gon, an 18-gon, a 20-gon, a 24-gon, and a 42-gon. In each of the five cases, find the other regular polygons that are needed.

C. With the additional conditions that tiles can only be placed edge to edge and all vertices are identical, there are only eleven possible regular polygon tilings. Find as many as you can.

D. Analyze the symmetries of the tilings that you found. Do they have lines of symmetry? Are all of those parallel to each other? If not, what angle do they make with each other? Do the tilings have centers of rotation symmetry? How are those arranged? Do the centers all have the same *n*-fold symmetry?