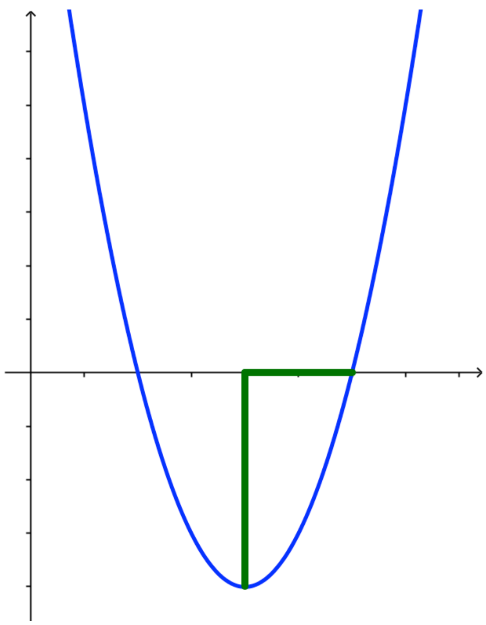
A Graphical Approach to the Quadratic Formula

This is an approach to the quadratic formula based on moving parabolas.

The solutions to $ax^2+bx+c = 0$ are the x-intercepts of $y = ax^2+bx+c$. The x-intercepts are equidistant from the axis of symmetry, at a distance we'll call d. We'll say that the coordinates of the vertex are (h,v). Thus, the intercepts are $h \pm d$.

0. Label this figure, using (h,v) for a certain point, and d for a certain line segment.



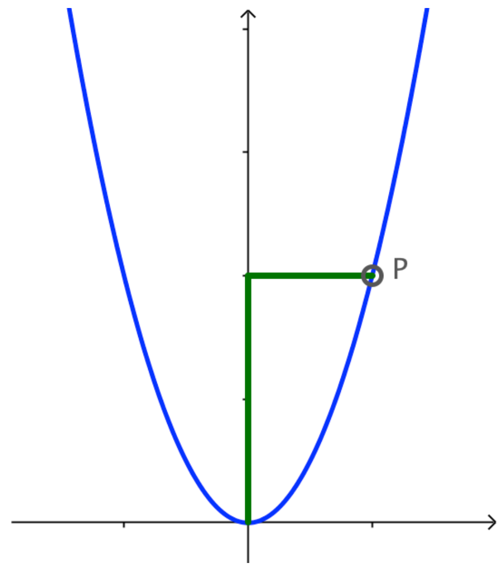
Review

1. What is h in terms of a, b, and/or c?
2. Use the answer to #1 to find v in terms of a, b, and/or c.
3. Discuss how the existence of the x-intercepts depends on the signs of v and a. (Four cases: both positive / $a>0$, $v<0$ / $a<0$, $v>0$ / both negative)
4. Use this to find one expression in terms of a, b, and c whose sign predicts the existence of the x-intercepts.

Derivation

If we find the value of d in terms of a, b, and c, we have essentially found a formula for the solutions of $ax^2+bx+c = 0$

Translate the two segments and the parabola together, so that the vertex is at the origin:



5. What is the equation for this parabola? Explain.
6. What are the coordinates of P in terms of d and v? Label P with its coordinates.
7. Write a formula for P's y-coordinate in terms of its x-coordinate.
8. Solve for d. (In other words, write a formula for d in terms of v, and then in terms of a, b, and c.)
9. Use this to derive the quadratic formula.

Notes and Answers

Prerequisite: this is best done following the lessons **Factored Form of Quadratic Equations**, **From Factored to Standard Form**, and **Moving Parabolas Around.** *It is intended as a lesson for teachers*, though it may in the end be more accessible to students than the usual proof of the quadratic formula by completing the square.

1. $h = -b/2a$

2. This is a tricky calculation. Many students are likely to need help. The result: $v = \frac{-b^2 + 4ac}{4a}$.

3 and 4. This is a way to discover or re-discover the discriminant. Basically, a and v must have opposite signs if we want to have x-intercepts. If a is positive, the parabola is a smile, and v needs to be non-positive for there to be solutions. So $b^2 - 4ac$ needs to be positive or zero. If on the other hand a is negative, the parabola is a frown, and v needs to be non-negative for there to be solutions. Again, $b^2 - 4ac$ needs to be positive or zero for there to be solutions. In short, there are solutions if and only if $b^2 - 4ac$ is non-negative. (This argument is not likely to be discovered by students. Help them out.)

5. The formula is simply $y = ax^2$, since translating a parabola does not change a, and the vertex is at the origin.

6. $(d, -v)$

7. Thus, $-v = ad^2$

8. $d = \pm\sqrt{\frac{-v}{a}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ after writing v in terms of a, b, c.

9. The intercepts are $h \pm d = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quadratic formula.