Computing Transformations

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Teacher Notes

These lessons provide the framework of an approach to computing geometric transformations in a precalculus level course, starting with complex numbers, and ending with matrices.

The lessons assume a fair amount of teacher explanation and class discussion, and should not be expected to teach themselves. The unit is based on material I taught at The Urban School of San Francisco, in my Space class, an advanced high school geometry elective. The course and this material evolved over a dozen iterations. See MathEducation.page/space.

Prerequisites

Students should have been introduced to complex numbers, and be familiar with their rectangular and polar representations. At The Urban School, that introduction happens towards the end of our Algebra 2 course. See our Complex Numbers unit at MathEducation.page/alg-2/complex.

Students should also have been introduced to matrices and matrix multiplication. I used the approach in the University of Chicago School Mathematics Project *Advanced Algebra* book.

Technology

Some lessons assume the use of the Texas Instruments TI-89 calculator, including some of its CAS capabilities. It is of course possible to do much of this work with other graphing calculators or computer software. See for example my attempt at a GeoGebra version, where you found this document. (If you create a version of this packet using some other technological support, I will gladly credit you and post it alongside this version.)

Review: Slope of Perpendicular Lines

This lesson reviews the “opposite reciprocal” result, but also proves it. It also includes a reminder of similar triangles. All of this will be needed in the subsequent pages.

Complex Numbers Basics (and Transformations)

This lesson combines a (perhaps too sketchy) review of complex numbers with an introduction to two key questions we address in this unit:
◊ How do we compute the coordinates of images of points under various transformations?
◊ How do we use composition of transformations we know how to compute, in order to figure out more challenging computations?

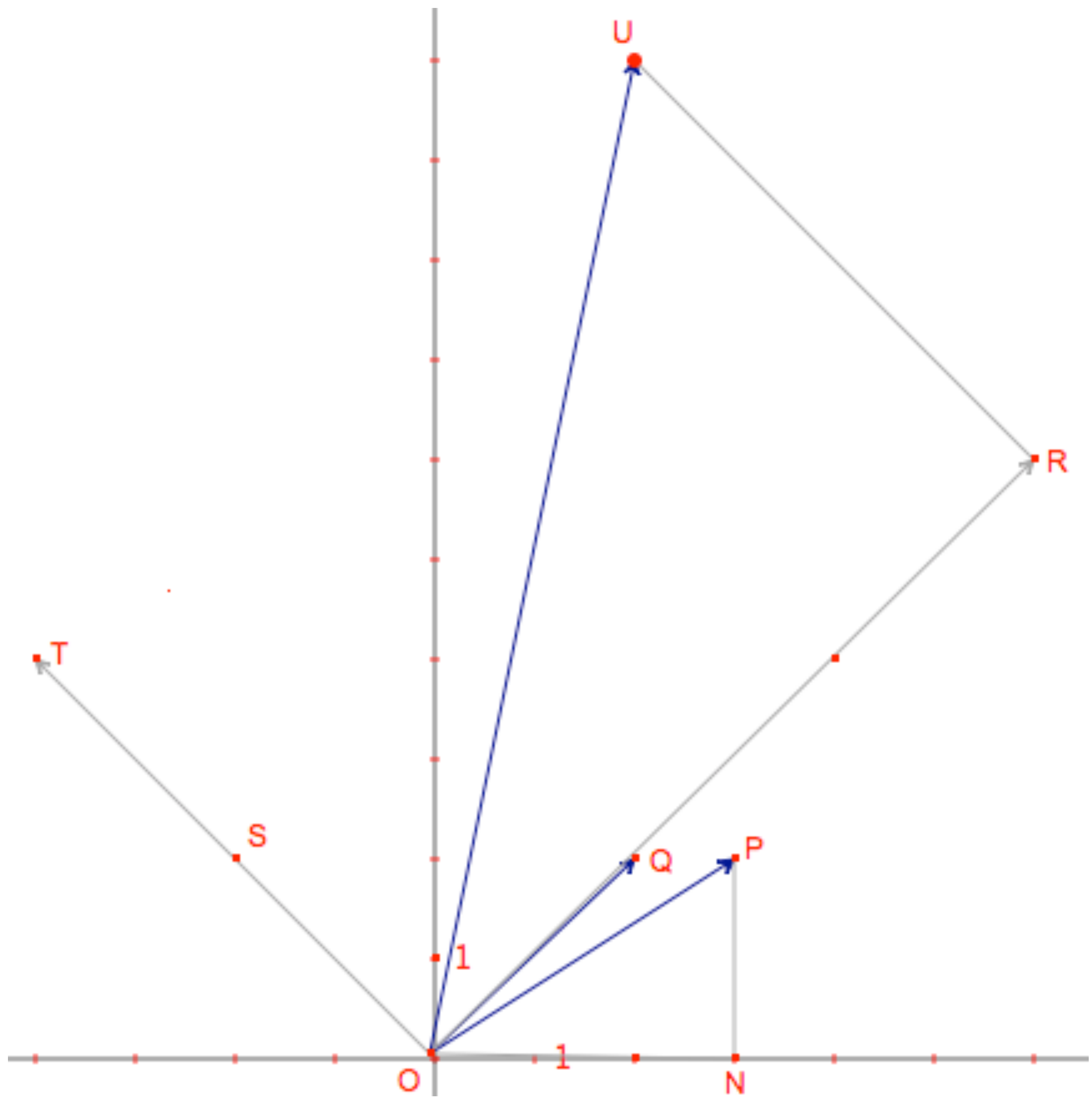
Note that #3 does not require knowing about multiplication in polar form. (This fact will be important later.)

#5. Multiply by $(1, \theta)$ -- a complex number on the unit circle

#6. Subtract p+qi, multiply by (1, θ), add p+qi. In other words, translate to the origin, rotate around the origin, translate back.

Complex Multiplication in Polar Form

This two-part lesson ends with a proof of the geometric interpretation of complex number multiplication. One way to teach it is to have the students work through the first page on graph paper, providing support as needed. Then, present the key ideas of the second page, culminating with the proof, without the worksheet. And finally have the students work through the second page themselves. This figure may be useful on a projector or interactive white board:



See also the corresponding GeoGebra file at mathed.page/space/mult-proof.ggb

The heart of the proof is SAS similarity between ΔONP and ΔORU. Avoid a circular argument! $U = P \cdot R$ follows from the argument in #2 of the first page which is generalized in #1 on the second page. The right angle at $\angle R$ follows from the argument involving slopes that we used in the "Complex Numbers" lesson.

Computing Any Isometry Using Complex Numbers

The five-step method for #7 is: translate so the y-intercept is at the origin, rotate so the line coincides with the x-axis, reflect, rotate back, translate back. Assuming $\theta = \tan^{-1}(2)$, this calculation will do it:

$$(\text{conj}(((3+2i) - 5i)\,(\cos(-\theta)+ i \sin(-\theta))))(\cos(\theta)+i \sin(\theta)) + 5i$$

Matrices

The remaining lessons on the one hand replicate the approach we used with complex numbers, using matrices, and on the other hand introduce increasingly efficient ways to use the TI-89 calculator to actually do the computations .

Having students themselves find the matrices for the transformations gives them a sense of control of this technique.

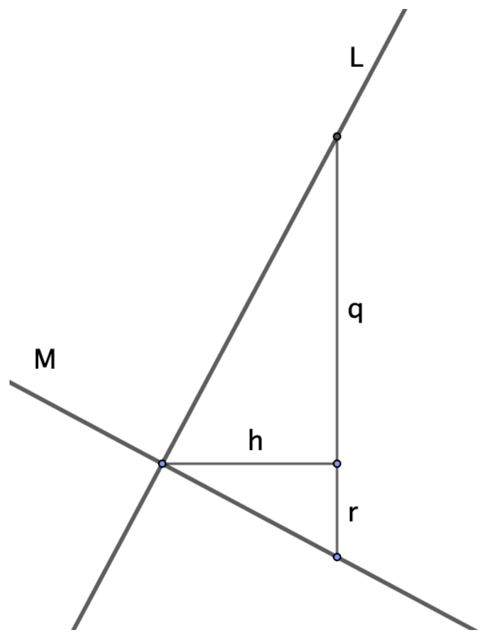
 Name:________________

Review: Slope of Perpendicular Lines

0. What are the three ways to show triangles are similar?

1. Consider two perpendicular lines L and M. What is the relationship between their slopes?

2. Use the slope triangles in the figure above to prove this. Hint: start by proving the triangles are similar. (Slope triangles by definition have perpendicular sides. The labels h, q, and r represent the lengths of the segments.)



3. Now prove the converse of this result. ("If the slopes of two lines are … then the lines are perpendicular.") Hint: you can do it by drawing two congruent right triangles.

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Complex Numbers Basics (and Transformations)

You can think of a complex number as a vector that starts at the origin.

Rectangular form: a+bi, with $i^2 = -1$.
a is the real part. b is the imaginary part.

Polar form: (r, θ)
r is known as the radius, the modulus, the magnitude, or the absolute value of the number.
θ is the angle or argument.

1. Use basic trig to convert from polar to rectangular form and vice versa. (Draw an example on graph paper to help you remember how to do this. Work out examples in each of the four quadrants.)

2. Complex addition: (a+bi)+(c+di).
   a. Rearrange the terms so that the real part is first, and the imaginary part last.
   b. Draw an example of this on graph paper, and explain how this addition works like vector addition.
   c. How would you use complex numbers to find the image of (a, b) by a *translation* (v, w)?

3. Complex multiplication: simple cases. For each example below, draw an example on graph paper.
   a. If k is a real number, distribute k(c+di). How does the resulting vector (the result of your calculation) compare with the original (c+di)? This transformation is called a ___________ with factor ____ and center at _______________.

   b. Distribute i(c+di). Write the result the usual way, with the real part first, and the imaginary part next. How does the resulting vector compare with the original? What is this transformation called?

   c. Distribute ki(c+di). Describe how the resulting vector compares with the original. This transformation does not have a name. It is the *composition* of the previous two.

In Math 3, you learned that $(r_1, \theta_1)\cdot(r_2, \theta_2) = (r_1r_2, \theta_1+\theta_2)$.

4. Put that in words: to multiply complex numbers in polar form…

5. How would you use complex numbers to find the image of (a, b) by a *rotation* of θ° centered at the origin?

6. How would you use complex numbers to find the image of (a, b) by a rotation of θ° centered at (p, q)?

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Complex Multiplication in Polar Form: A Specific Example

In Math 3, you learned that $(r_1, \theta_1)\cdot(r_2, \theta_2) = (r_1r_2, \theta_1+\theta_2)$. We will use geometry to prove this, starting from the definition of i as the square root of –1.

0. Multiplying a complex number by i is the same as a ______________ around _____________

We will start by analyzing a specific example: $(2+i)(3+4i)$

1. a. Draw these two complex numbers as vectors on graph paper.
   b. Label the origin as O.
   c. Label the point $2+i$ as P, and $3+4i$ as Q.
   d. Label the point directly below P on the x-axis N.

2. Distributing, we see that $P\cdot Q = (2+i)Q = 2Q + iQ$
   a. Find the points for 2Q and iQ on your graph. Label them R and T.
   b. Find the point for $2Q+iQ$, and label it U.

3. Explain: $P\cdot Q=U$

We'd like to show that the magnitude of U is the product of the magnitudes of P and Q, and that it's angle is the sum of their angles. We will do this with the help of triangles ΔOUR and ΔOPN.

4. Using only the information about how we created U and R, show that ΔOUR and ΔOPN are similar. What is the scaling factor?

5. Show that $|U|=|P|\cdot|Q|$

6. Show that $\angle NOU = \angle NOP+\angle NOQ$

If you understand this so far, you're ready to generalize.

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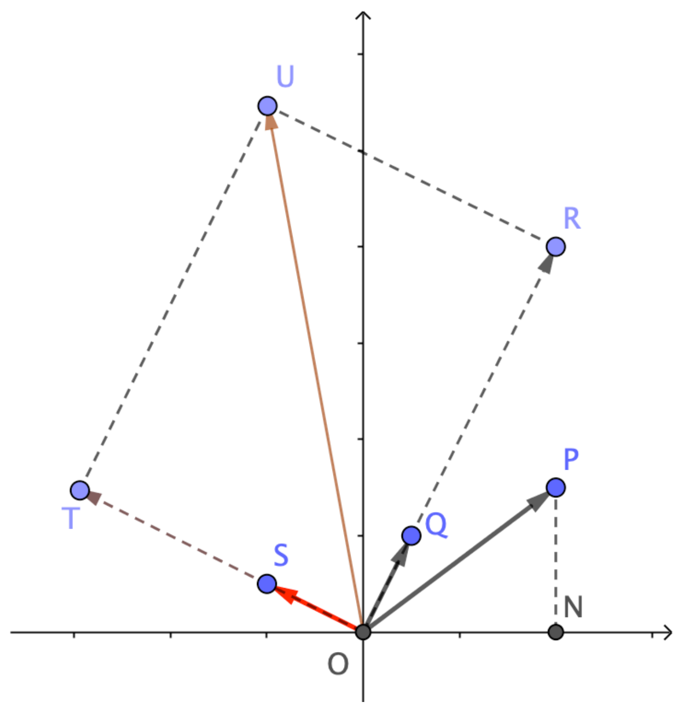
Complex Multiplication in Polar Form: Generalizing

This time we will go through the argument with a generic figure. P=a+bi, Q=c+di, and the other points are defined as in the figure below.

1. Explain this: (a+bi)(c+di) = a(c+di) + bi(c+di).

Another way to write the above is:
(a+bi)Q = aQ + biQ if Q is the complex number (c+di).



By answering the next few questions, you will see that this figure is an illustration of the above equation.

2. On the figure, what point represents:
   a. aQ
   b. iQ
   c. biQ
   d. aQ + biQ

3. Explain why U = P·Q, using #2.

Up to this point, we worked with the rectangular form of P, Q, and U. To prove the result we are after, we will now switch to polar form, and use similar triangles in the figure above.

4. Say that P = (r_1, θ_1), Q = (r_2, θ_2), and U = (r, θ). Label the figure accordingly.

5. Prove that ΔONP is similar to ΔORU. What is the scaling factor?

6. Prove that $r = r_1 r_2$.

7. Prove that $\theta = \theta_1+\theta_2$.

Therefore: $(r_1, \theta_1)\cdot(r_2, \theta_2) = (r_1 r_2, \theta_1+\theta_2)$

QED

 Name:__________________

Computing Any Isometry Using Complex Numbers

Calculator shortcuts:

- ◊ Converting from rectangular to polar: for example 3+4i → Polar (in the Catalog under P)
- ◊ Converting from polar to rectangular: it's simply $r\cdot\cos(\theta)+r\cdot\sin(\theta)\cdot i$, or for example (3 ∠ 45) → Rectangular (in the Catalog under R) (∠ is 2nd EE)

1. Make sure you know how to compute the image of a point (a,b) under any translation or rotation, using complex numbers.
   a. Translation by a vector (v,w)
   b. Translation by a vector (r,θ)
   c. Rotation by θ degrees around the origin.
   d. Rotation by θ degrees around a point (p,q)

What we are missing is a method for computing reflections.

2. The image of (a,b) after reflection in the x-axis is ___________.

3. The image of a+bi after reflection in the x-axis is ___________.

This image is called the conjugate of a+bi, which the TI-89 will return if you enter **conj(a+bi)**.

Three-step method: We will find the image of (a,b) in a line y=mx, by rotating so the line lies on the x-axis, then we'll reflect across the x-axis, then we'll rotate back.

4. What is the angle between y = 2x and the x-axis?

5. What is the reflection of the point (3,2) across the line with equation y=2x?

6. Now let us reflect (3,2) in the line y=x.
   a. Predict the coordinates of the image
   b. Check whether the three-step method gives you that answer.

7. Work out a strategy (five steps) to reflect (3,2) across the line with equation y = 2x+5. Find the coordinates of the image.

8. Work out a strategy to glide-reflect (3,2) with mirror y=x+4 and vector (5,5)

You can now compute the results of any isometry! However this method has shortcomings:
◊ It is tricky to keep track of parentheses and order of operations.
◊ It is inconvenient when calculating the image of a polygon, as the calculation has to be done for each vertex.
◊ It does not generalize to three dimensions.
All of those concerns are addressed by using matrices. In fact, all computer animation is done by matrix calculations of geometric transformations. This is what we will learn next.

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Computing Some Transformations with Matrices

We will represent points using a vertical (2 by 1) format: $\begin{bmatrix} x \\ y \end{bmatrix}$. An n-gon will be represented by a 2 by n matrix. For example: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ represents a right triangle.

1. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a dilation centered at the origin, with factor k.

2. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in the y-axis.

3. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in the y=x line.

4. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a 180° rotation around the origin.

In a previous lesson, you found the formula for the image of a point (x,y) under a rotation of θ° around the origin.

5. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a rotation of θ° around the origin.

In a previous lesson, you found a way to use the complex conjugate as part of a three-step process to reflect a point across any line through the origin.

6. Find a sequence of three matrices L, M, N such that $N \cdot M \cdot L \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in the y=2x line. Note that the matrices are written from right to left in the multiplication.

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Matrices on the Calculator

Your calculator can do matrix multiplication.

To enter a matrix, enclose it in square brackets (2nd comma, and 2nd ÷). Within that, each row is enclosed in square brackets, and the items on the row are separated by commas.

For example, to enter the multiplication $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, type: [[1,0][0,-1]]x[[2][3]]

1. Check that the above multiplication gives the expected result.

If, for example, you want to use sin 56° in a matrix, just enter sin(56). You don't have to get a decimal for it.

2. Use matrix multiplication to find the image of (2, 3) after a rotation of 22° around the origin.

Matrices can be saved in memory.

3. The next three problems will use a 33° rotation. Create the matrix for it and use STO to store it as **r**.

We don't yet know how to use matrix multiplication for translations, so you'll have to do translations yourself in the next two problems.

4. Use matrix multiplication to find the image of (2, 3) after a rotation of 33° around (5, 4)
5. Use matrix multiplication to find the image of (5, 4) after a rotation of 33° around (2, 3)

Matrices allow you to transform many points in a single calculation.

For example, to rotate (9, 4), (7, 5), and (8, 6) 33° around the origin, you can do r$\cdot \begin{bmatrix} 9 & 7 & 8 \\ 4 & 5 & 6 \end{bmatrix}$

6. What are the images of the three points?

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Matrices on the Calculator: Shortcuts

It is tedious to enter matrices over and over. Here is a shortcut, using STO.

[[cos(t),-sin(t)][sin(t),cos(t)]] → ro
[[1,0][0,-1]] → rx

You will not need to enter these matrices again.

1. What do these matrices do? You should recognize them without looking at your notes.

2. Explain the calculator's response if you enter
   ro | t=45
   ro | t=60

3. How would you use matrix multiplication to reflect a point (x_0,y_0) around a line that makes an angle θ with the positive x-axis?
   a. Say this in words, not symbols

   b. Write the matrix multiplication, remembering that you go from right to left.

4. Test your answer with the matrices we entered, using 45° since we know what the result of that reflection would be. Here is how you would enter this:
   (ro | t=45)*rx*(ro | t=-45)[[x0][y0]]
   Make sure you got the answer you expected.

5. If a line has equation y = 3x, what angle does it make with the positive x-axis?

6. What is the image of (4, 5) across the line y = 3x?

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Matrices and Translation

The shortcoming of using matrix multiplication in the way we have been is that it does not work for translations. This problem is solved if we represent points as 3 by 1 matrices, like this: $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

1. Find a 3 by 3 matrix M such that $M\cdot\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a translation by vector (v,w).

2. Find 3 by 3 matrices for
   a. Reflection in the x-axis
   b. Rotation around the origin by an angle θ.

3. Find a sequence of three matrices L, M, N such that $N\cdot M\cdot L\cdot\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a θ° rotation around a point (p,q). Remember that the matrices are written from right to left in the multiplication.

4. Find a sequence of five matrices J, K, L, M, N such that $N\cdot M\cdot L\cdot K\cdot J\cdot\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a reflection across the line y=mx+b.

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Better Calculator Shortcuts

Setup

1. To create an easy-to-use *function* that will output a 3 by 3 translation matrix, use **Define**, which is in the **F4** menu:
   Define tr(v,w)=[[1,0,v][0,1,w][0,0,1]]
   From now on, **tr(3,4)**, for example, will output the translation matrix for vector (3,4).

2. Use the same method to make functions for:
   a. a 3 by 3 matrix for a rotation of t° around the origin -- **ro(t)**. (You may need to delete the matrix **ro** first.)
   b. a 3 by 1 matrix for a point with coordinates (a,b) -- **pt(a,b)**

3. For reflection across the x-axis, you don't need a function, as it is always the same matrix. If you don't yet have it, use **STO** to store the 3 by 3 matrix under the name **rx**.

4. For convenience, you should also store a 3 by 1 matrix for (x,y) under the name **xy**.

Practice

5. To rotate (2,3) 53° around (4,3), do this:
   tr(4,3)*ro(53)*tr(-4,-3)*pt(2,3)

6. To find a matrix for rotation of any point (x,y) t° around (a,b):
   tr(a,b)*ro(t)*tr(-a,-b)*xy

7. Reflect (2,3) across the line y=x+4
   a. "manually" on graph paper
   b. using matrices on the calculator (you need five matrices in reverse order!)
   Hopefully you get the same answer both ways.

Polygons

8. Store a 3-rows, n-columns matrix (n≥3) for a polygon, as **po**. I recommend: $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

To draw your polygon, you will need to get the **dr** program into your calculator.

9. You can draw your polygon by entering: **dr(po)**

10. You can draw images of your polygon by entering expressions like **dr(ro(45)*po).** Make an interesting design!

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Review: From Complex Numbers to Matrices

A point in the plane can be thought of as a ____________ number, just like a point on a number line can be thought of as a ________ number.

1. Write the following three famous points in a+bi (rectangular) form:
   a. (1, 45°)
   b. (1, 60°)
   c. (1, 90°)
   For the purpose of the following exercises, call these points respectively s, t, and i.

2. s, t, and i are all on a certain geometric figure. What am I referring to?

3. Using your calculator, or not, compute the following, and show the results on graph paper:

a. s+t	b. t+i	c. s+i
d. s·t	e. t·i	f. s·i

4. Describe the results of the computations above, using the words *translation* or *rotation*. (There are two ways of doing this for each example.)

5. Explain how to use complex numbers to translate a point (x, y) by a vector (v, w)

6. Write the coordinates of the point (1,θ) in a+bi form.

7. Explain how to use complex numbers to rotate a point (x, y) by an angle θ around the origin.

8. Use the answer to the previous problem to explain where the rotation matrix comes from.

```
dr(p)
Prgm
For i,1,colDim(p)-1
Line p[1,i],p[2,i],p[1,i+1],p[2,i+1]
EndFor
Line
p[1,1],p[2,1],p[1,colDim(p)],p[2,colD
im(p)]
EndPrgm
```