CHAPTER 11

A spiral staircase

Coming in this chapter:

Exploration On graph paper, you want to go from (0, 0) to (p, q), where p and q are positive whole numbers. If you travel only up and to the right, following graph paper lines, how many ways are there to do it? If you travel in a straight line, how many graph paper squares do you cross?

INTERPRETING RATIOS

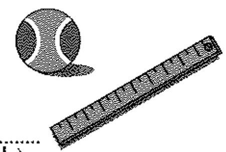
LESSON 11.1

Sums of Geometric Sequences

You will need:

a ball

yardstick (or meterstick)



THE BOUNCING BALL

When you drop a ball, it bounces back, but not quite to the height from which you dropped it.

1. Do an experiment in which you drop a ball from various heights and see to what height it bounces back. Use a yardstick or meterstick to make your measurements. Make a table like this.

Dropped from	Bounced to	Ratio	Difference
——	——	——	——

2. As you vary the height, what remains closer to constant, the ratio or the difference?

For a certain "ideal" ball, the bounce-height to drop-height ratio (or *bounce ratio*) is consistently 0.8. The ball is dropped from a height of two meters.

3. a. How high does it bounce on the first, second, and third bounces?

b. How many bounces until it bounces to fewer than 80 centimeters?

c. How many bounces until it bounces to fewer than 10 cm?

4. What is the total distance traveled by the ball (both down and up) if someone catches it at the top of its bounce after:

a. 2 bounces? b. 20 bounces?

5. Make a guess about the total distance traveled by the ball after 200 bounces. Justify your guess.

USING SYMBOLIC NOTATION

Say the bounce ratio is r. Then we have:

$$\frac{\textit{bounce height}}{\textit{drop height}} = r$$

Or: *bounce height* $= r \cdot$ *drop height*

Assume that the initial drop height is H.

6. How high does the ball bounce on the first, second, third, and fourth bounces? Express your answers in terms of H and r.

To analyze the problem of the total distance traveled, it is easier to separate the upwards and downwards motions. First find the downwards distance traveled in the first four bounces.

$$D_4 = H + Hr + Hr^2 + Hr^3$$

As you see, the terms of the sum form a *geometric sequence* having first term H and *common ratio r*.

7. Write an expression for D_6 the downwards distance traveled in the first six bounces.

8. What is the last exponent in the expression for the downwards distance traveled in the first n bounces? Explain why the exponent is not the same as the number of bounces.

9. Write an expression for the upwards distance traveled in:

a. the first four bounces, U_4;

b. the first six bounces, U_6.

10. What is the last exponent in the expression for the upwards distance traveled in the first n bounces? Why does this differ from the expression for the downwards distance?

FINDING THE SUM

Here is a shortcut for calculating the sum of a geometric sequence. We will use the example of the ideal ball having bounce ratio 0.8, dropped from a height of two meters, and caught at the top of its fourth bounce. First write the downwards motion.

Eq. 1: $D_4 = 2 + 2(0.8) + 2(0.8)^2 + 2(0.8)^3$

Do not calculate the sum! You will soon see why.

Multiplying both sides by 0.8, we get:

Eq. 2: $D_4(0.8) = 2(0.8) + 2(0.8)^2 + 2(0.8)^3 + 2(0.8)^4$

Subtracting one equation from the other:

Eq. 1-Eq. 2: $D_4 - D_4 \cdot (0.8) = 2 - 2(0.8)^4$

11. Explain why there are so few terms after subtracting.

12. *Solve* for D_4. (Hint: Factor, then divide.)

13. Use this *multiply-subtract-solve* technique to find U_4. You found an expression for U_4 in problem 9.

14. What is the total distance traveled by the ball in four bounces?

When adding only four terms, the multiply-subtract-solve technique is not much of a shortcut. However, when adding large numbers of terms, it is extremely convenient. For example, for 20 bounces, you would start by writing:

$D_{20} = 2 + 2(0.8) + ... + 2(0.8)^{18} + 2(0.8)^{19}$

15. Explain why in this case the last terms do not contribute very much to the sum.

16. Use the multiply-subtract-solve technique to check the correctness of your answers for problems 4b and 5.

OTHER BOUNCE RATIOS

17. What is the total distance traveled in 200 bounces by a ball having the following bounce ratios, after being dropped from a height of two meters?

a. a super-ball, having bounce ratio 0.9

b. a flat basketball, having bounce ratio 0.3

18. Repeat problems 3-5 for a real ball. (First, you must find the bounce ratio, perhaps by averaging the ratios you found in problem 1.) Verify your predictions for problem 3 with experiments.

An absent-minded professor invents a hyper-ball having a bounce ratio of 1.1.

19. Repeat problems 3-5 for the hyper-ball.

20. Repeat problems 3-5 for a defective hyper-ball having a bounce ratio of only 1.

21. Summary Summarize what you learned about the sum of geometric sequences.

a. Explain the multiply-subtract-solve method. (What does one multiply by? What does one subtract? What does one solve for, and how?)

b. What is the effect of the common ratio on the sum? (What if r is less than 1? What if it is equal to 1? What if it is greater than 1?)

22. Generalization Use the multiply-subtract-solve technique for each sum S.

a. $S = a + ar + ar^2 + ... + ar^{n-1}$

b. $S = a + ar + ar^2 + ... + ar^n$

DISCOVERY *FOUR NUMBERS*

23. a. Replace each box with one of the numbers: 1, 2, 3, 4. (Use each number exactly once.)

$$\frac{\square}{\square} + \frac{\square}{\square}$$

How many possible arrangements are there?

b. Which arrangement gives the smallest sum? What is the smallest sum?

c. Which arrangement gives the largest sum? What is the largest sum?

d. Are the arrangements that give the smallest and the largest answer *unique*? That is, is there only one arrangement that gives the same sum?

24. Repeat problem 23 for $\frac{\square}{\square} - \frac{\square}{\square}$, this time finding the arrangements that give the smallest and the largest difference. How are the smallest and the largest difference related? Explain.

25. Repeat problem 23 for $\frac{\square}{\square} \cdot \frac{\square}{\square}$, this time finding the arrangements that give the smallest and the largest product. How are the smallest and the largest product related? Explain.

26. Repeat problem 23 for $\frac{\square}{\square} \div \frac{\square}{\square}$, this time finding the arrangements that give the smallest and the largest quotient. How are the smallest and the largest quotient related? Explain.

27. Choose four numbers a, b, c, d such that $a < b < c < d$. Repeat problems 23-26 for these numbers. Compare your answers with other students' answers. Were you able to use the answers from problems 23-26 to help you?

28. **Report** Write a report summarizing your findings in problems 23 through 27. Describe the strategies you used for finding the smallest and the largest values. Explain why you were sure that they were the smallest and the largest.

LESSON 11.2

Decimals and Fractions

WRITING FRACTIONS AS DECIMALS

1. How do you convert a fraction to a decimal number? Give examples.

When converting fractions to decimals, sometimes you get a *terminating* decimal like 3.4125, and sometimes you get a *repeating* decimal, like 7.8191919.... This last number is often written $7.8\overline{19}$.

Problems 2 and 3 are easier if you work with lowest-term fractions.

2. Exploration For what fractions do you get a repeating decimal? Does it depend on the numerator or the denominator? (Hint: Pay attention to the prime factorization of the numerator and the denominator.)

3. Exploration For repeating decimals, is there a pattern to the number of digits in the repeating part? What is the longest possible repeating string for a given denominator? (Hint: Use long division rather than a calculator to explore this.)

4. Explain why the decimals obtained as a result of a division *must* repeat or terminate.

5. Explain why some calculators give a decimal that does not seem to repeat for 2/3: 0.6666666667.

WRITING DECIMALS AS FRACTIONS

Example: 3.4125 can be converted to a fraction by multiplying it by 10^4, which gets rid of the decimal, and then dividing by 10^4, which gets us back to the original number.

$$\frac{34{,}125}{10{,}000}$$

6. Convert these decimals to fractions.

a. 6.0 b. 3.2

c. 0.015 d. 3.41

The case of repeating decimals is more difficult. Take $7.8\overline{19}$. Clearly, it is greater than 7.81 and less than 7.82. So it is between 781/100 and 782/100.

To find a single fraction it is equal to, we can rewrite it as:

$$\begin{aligned} &7.80\overline{19} \\ &= 7.8 + 0.0\overline{19} \\ &= 7.8 + 0.019 + 0.00019 + 0.0000019 + \ldots \end{aligned}$$

Observe that:

$$0.00019 = 0.019(0.01)$$
$$0.0000019 = 0.019(0.01)^2$$

7. Write the next term in the sum as a decimal, and as a product of 0.019 and a power of 0.01.

As you see, $7.8\overline{19}$ is the sum of 7.8 and a geometric sequence with first term 0.019 and common ratio 0.01. The sum of the first three terms of the geometric sequence can be written:

$$S = 0.019 + 0.019(0.01) + 0.019(0.01)^2$$

Multiply both sides by 0.01:

$$S(0.01) = 0.019(0.01) + 0.019(0.01)^2 + 0.019(0.01)^3$$

Subtract:

$$S(1 - 0.01) = 0.019 - 0.019(0.01)^3$$

Solve:

$$S = \frac{0.019 - 0.019(0.01)^3}{0.99}$$

Multiplying numerator and denominator by 1000:

$$S = \frac{19 - 19(0.01)^3}{990}$$

$$7.8\overline{19} = 7.8 + S$$

$$= 7.8 + \frac{19 - 19(0.01)^3}{990}$$

$$= \frac{7.8(990) + 19 - 19(0.01)^3}{990}$$

So

$$= \frac{7741 - 19(0.01)^3}{990}$$

$$= \frac{7741 - 0.000019}{990}$$

The sum is very close to 7741/990.

8. Use the multiply-subtract-solve technique to add:

a. the first 4 terms;

b. the first 5 terms.

9. 🔑 The numerator differs from 7741 by $19(0.01)^n$ if we add up the first n terms. Explain.

If we use large values for n, we find that the sum can get as close to 7741/990 as we want. (Even with fairly small values of n, the sum of the first n terms differs from 7741/990 by a *very* small number.) Mathematicians say that the whole infinite sum *converges* to 7741/990, and they agree that we can write an equality:

$$7.8\overline{19} = 7741/990.$$

10. Check that this equality is correct, by converting the fraction back to a decimal.

A quick way to find the fraction is to use the multiply-subtract-solve technique on the decimal itself:

$$R = 7.8191919...$$
$$0.01\,R = 0.0781919...$$

Subtract:

$$R - 0.01R = 7.8191919... - 0.0781919...$$
$$(1 - 0.01)R = 7.819 - 0.078$$

(Notice that the infinite sequence of 19s disappeared.)

$$0.99R = 7.741$$

$$R = \frac{7.741}{0.99} = \frac{7741}{990}$$

11. Convert to a fraction.

a. $0.\overline{65}$ b. $4.\overline{321}$

RATIONAL NUMBERS

Definition: A *rational number* is a number that can be written as a fraction having an integer numerator and denominator.

Examples: 7, 0.5, and -0.66666... are rational numbers, because they can be written as 7/1, 1/2, and -2/3.

Show that the following numbers are rational.

12. a. 0.3

b. 0.3333...

13. a. 0.142857

b. $0.\overline{142857}$

14. a. 0.0909090...

b. 0.9090909...

15. a. 0.1111111...

b. 0.2222222...

16. 🔑 Mathematicians believe that 0.99999... = 1. Explain why.

LESSON 11.3

Stairs and Squares

You will need:
- graph paper
- geoboard

STAIR SAFETY

In most houses, stairs have a riser (or rise) of eight inches and a tread (or run) of nine inches. However, safety experts claim that such stairs are the cause of many accidents. They recommend what they call 7/11 stairs: a riser of seven inches, and a tread of eleven inches.

1. What are the slopes of the stairs described in the previous paragraph? (Express the answer as a decimal.)

2. If a staircase makes a vertical rise of about nine feet from one floor to the next, how much horizontal distance does it take
a. for 8/9 stairs?
b. for 7/11 stairs?

3. Why do you think 8/9 stairs are more common?

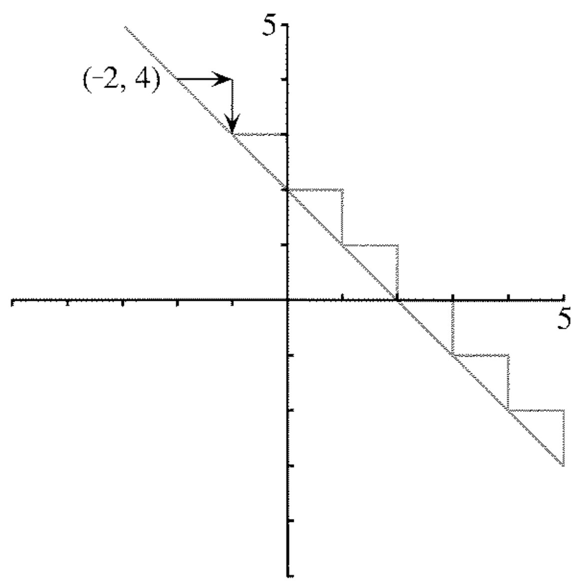
4. **Exploration** Donna wants to build a staircase that is less steep than an 8/9 staircase would be, but that does not take up as much horizontal space as a 7/11 staircase would. What are the possibilities for the riser and tread of Donna's stairs? Make the following assumptions:
- The riser and tread must each be a whole number of inches.
- The riser should be between six and nine inches, inclusive.
- The tread should be between eight and twelve inches, inclusive.

Express your answer numerically and graphically.

STAIRS ON LINES

To build a staircase on the graph of a line:
a. Sketch the graph.
b. Find the coordinates of a point on the graph. Call this point the starting point.
c. Find two numbers (the rise and the run) such that if you draw a step having those dimensions, you end up on the line.

The figure shows a staircase for the line $y = -x + 2$.



The starting point is (-2, 4), the rise is -1, and the run is 1.

5. Create a staircase for the same line, using a different starting point and a different rise and run.

6. Create *two* staircases for each line. They must have a different starting point and a different rise and run.
a. $y = -4 + 3x$
b. $y = -0.5x$
c. $y = 9$

d. $y = \frac{6x - 7}{8}$

e. $y = -2(x - 3)$

7. Find a rise and a run for a staircase connecting the following pairs of points:

a. (3, -5) and (2, 2.5)

b. (-3, 5) and (2, 2.5)

8. A staircase having the given rise and run starts at the given point. What is the equation of the corresponding line?

a. rise = 4, run = 6, point = (-3, 6)

b. rise = -2, run = -3, point = (0, 8)

LATTICE POINTS AND FRACTIONS

Definition: A *lattice point* is a point on the Cartesian plane having integer coordinates.

Examples: (2, 3) is a lattice point, but (4.5, 6) is not.

9. The graph of each of the following equations is a line through the origin. Find two other lattice points on each line.

a. $y = 7x$ b. $y = \frac{2}{3}x$

c. $y = 4.5x$ d. $y = 6.78x$

If a line passes through the origin and the lattice point (9, 8), it will also pass through the lattice points $(9n, 8n)$ for all integer values of n.

10. If a line passes through the origin and the point (2.4, 3.6),

a. what are the lattice points on the line that are closest to the origin?

b. what is a general description of all the lattice points on the line?

c. what is the equation of the line?

11. Do all lines through the origin pass through another lattice point sooner or later? Discuss.

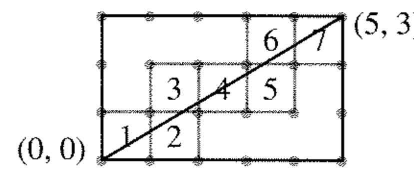
Generalizations

12. What is the slope of a line that passes through the origin and a lattice point (p, q), where $p \neq 0$?

13. Describe the lattice points on the line $y = (p/q)x$, where $p \neq 0$.

GEOBOARD DIAGONALS

If you connect (0, 0) to (5, 3) with a straight line, you go through seven unit squares.



14. Exploration If you connect (0, 0) to (p, q) with a straight line, how many unit squares do you go through? Experiment and look for patterns. (Assume p and q are positive whole numbers.) Keep a record of your work.

Definition: A *lattice line* is a line having equation $x = b$ or $y = b$, where b is an integer.

The following problems are about the diagonal connecting (0, 0) to (p, q). Give answers in terms of p and q.

15. a. How many horizontal lattice lines does it cross? (Look at some specific cases and make a generalization. Do not guess.)

b. How many vertical lattice lines does it cross?

16. How many lattice points does it cross,

a. if the greatest common factor of p and q is 1?

b. if the greatest common factor of p and q is n, where $n > 1$? (Experiment and reason. Do not guess.)

17. The diagonal starts in the first unit square, then every time it crosses a lattice line it enters a new square.

a. If it crosses no lattice points, how many squares does it go through altogether?

b. If it crosses n lattice points, how many squares does it cross?

18. Report How many squares do the diagonals of geoboard rectangles go through? Write an illustrated report, including examples.

DISCOVERY **SLOPE RELATIONSHIPS**

Lines
parallel
perpendicular
symmetric across horizontal line
symmetric across the line $y = x$
symmetric across vertical line

Slopes
opposite
opposite of reciprocal
reciprocal
reciprocal of opposite
same

The first column shows possible relationships between two lines. The second column shows possible relationships between the slopes of two lines.

19. Project Experiment to find out if it is possible to match relationships in the first column with relationships in the second column. (For example, parallel lines have the same slope.) Support your answers with examples, sketches, and explanations.

LESSON 11.4

Irrational Numbers

In Lesson 2 you learned how to show that any terminating or repeating decimal can be converted to a fraction. In other words, you know how to show that terminating or repeating decimals are rational numbers.

If a decimal is neither repeating nor terminating, it represents an *irrational number* (one that is not rational).

For example, the number

$$0.010110111011110111110...,$$

created by inserting one, two, three, ... 1's between the 0's, never ends or repeats. Therefore it cannot be written as a fraction, because if it were, it would have to terminate or repeat.

1. Create an irrational number that is
   a. greater than 1 and less than 1.1;
   b. greater than 1.11 and less than 1.12.

While most numbers we deal with every day are rational, and even though there is an infinite number of rational numbers, mathematicians have proved that most real numbers are irrational.

$\sqrt{2}$ and $\sqrt{3}$ are familiar examples of irrational numbers. They cannot be written as a fraction having whole number numerators and denominators. In order to prove this, we will need to review prime factorization.

PRIME FACTORIZATION

Every whole number can be written as a product of prime factors.

Example: $990 = 99 \cdot 10$
$= 9 \cdot 11 \cdot 2 \cdot 5$
$= 2 \cdot 3 \cdot 3 \cdot 5 \cdot 11$

Note that 990 has a total of five prime factors. (Three is counted twice since it appears twice.)

2. Start the factorization of 990 by writing $990 = 3 \cdot 330$. Do you get the same prime factors?

3. Start the factorization of 990 a third way. Do you get the same prime factors?

Each whole number greater than 1 has *only one* prime factorization. Find it for the following numbers:

4. 12

5. 345

6. 6789

7. Find the prime factorization of several perfect squares. Try to find one having an odd number of prime factors.

Take the numbers 6 and 8. We have

$$6 = 2 \cdot 3 \text{ and } 8 = 2^3.$$

Six has two prime factors, an even number. Eight has three prime factors, an odd number. When we square them, we get:

$$6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$$
$$8^2 = (2^3)^2 = 2^6$$

8. Explain why any perfect square *must* have an even number of prime factors.

9. Explain why any number that is equal to twice a perfect square *must* have an odd number of prime factors.

THE SQUARE ROOT OF TWO

This section explains why $\sqrt{2}$ is not a rational number. The way we are going to do this is to show that if it were, it would lead to an impossible situation. This is called proof by contradiction.

If p and q were nonzero whole numbers and we had

$$\frac{p}{q} = \sqrt{2}$$

It would follow that

$$\left(\frac{p}{q}\right)^2 = \left(\sqrt{2}\right)^2$$

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

10. Explain each step in the previous calculations.

11. Explain why p^2 must have an even number of prime factors.

12. Explain why $2q^2$ must have an odd number of prime factors.

13. Explain why p^2 cannot equal $2q^2$.

We conclude that there can be no whole numbers p and q such that $\sqrt{2} = p/q$, and therefore $\sqrt{2}$ is irrational.

14. Use the same method to show that $\sqrt{3}$ is irrational.

15. Show why the method does not work to prove that $\sqrt{4}$ is irrational.

16. Does the decimal expansion of $\sqrt{2}$ terminate or repeat?

17. Does the line $y = \sqrt{2}x$ pass through any lattice points?

18. Do all lines through the origin eventually pass through a lattice point? Discuss.

19. Research π is probably the world's most famous irrational number. Find out about its history.

DISCOVERY *SUM FRACTIONS*

20. Find two lowest-term fractions having different denominators whose sum is 8/9.

DISCOVERY *COMPARING COUPONS*

21. Which is a better deal, 15% off the purchase price, or \$1 off every \$5 spent? Make a graph that shows how much you save with each discount, for various purchases from \$1 to \$20. Write about your conclusions.

THINKING WRITING

11.A Nested Squares

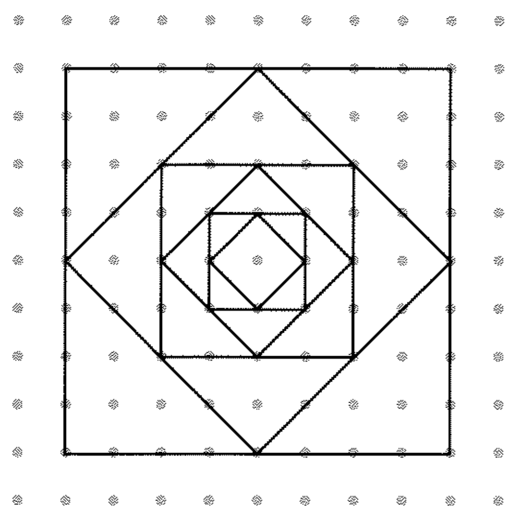
You will need:

geoboard

and/or dot paper

1. Using your geoboard or dot paper, make an 8-by-8 square. Calculate its area and perimeter.

2. Now make a square that is nested in the original square, like in the diagram. Its vertices should be the midpoints of the sides of the original square. Find its area and perimeter.



3. Continue the process, making smaller and smaller nested squares. As you work, extend and complete a table like the following one up to Square #5. When the numbers involve square roots, write them in simple radical form.

Square #	Area	Side	Perimeter
1	64	8	32

4. Look for a pattern in each of the columns. Describe the patterns for the
a. areas;
b. sides;
c. perimeters.

5. Use the pattern you found in problem 4. For the 10^{th} nested square, find
a. the area;
b. the side;
c. the perimeter.

6. Repeat problem 5 for the n^{th} nested square.

7. For the first ten squares, what is the sum of:
a. the areas;
b. the sides;
c. the perimeters.

8. Repeat problem 7 for the first n squares.

9. With larger and larger values of n, the sums get closer and closer to a certain number. What is that number for:
a. the areas?
b. the sides?
c. the perimeters?

10. Report Write a report on nested squares.

LESSON

11.5 Dice Games

You will need:

dice 

TWO GAMES

1. **Exploration** Play these two games with another person. To play a game, roll a pair of dice 20 times. After each roll, add the numbers on the uppermost faces. Keep track of how many rolls each player wins. (See below.) Whoever wins the most rolls, wins that game.

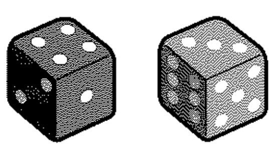
   Game One: If the sum is 3, 5, 7, 9, or 11, Player A wins. If the sum is 2, 4, 6, 8, 10, or 12, Player B wins.

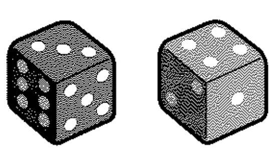
   Game Two: If the sum is 5, 6, 7, 8, or 9, Player A wins. If the sum is 2, 3, 4, 10, 11, or 12, Player B wins.

   For each game, who wins more often? Why?

TWO-DICE SUMS

If you roll a red die and a blue die, there are many possible outcomes. We will use (4, 3) to refer to the outcome in which 4 dots appear uppermost on the red die and 3 dots appear uppermost on the blue die. Likewise (3, 4) refers to 3 on the red die and 4 on the blue die.

(4, 3) 

(3, 4) 

Both of the outcomes in the figure show a sum of seven.

2. Copy and extend this table to show all possible two-dice sums. For each sum, list all the possible ways it can be obtained, and give the total number of ways. The sums of 2 and 7 have been done to get you started.

Sum	2	...	7	...	12
Possible ways	(1, 1)	...	(1, 6) (2, 5) (3, 4) (4, 3) (5, 2) (6, 1)	...	...
# of ways	1	...	6	...	...

3. Which sums have the most ways of occurring? Which sums have the fewest ways of occurring?

4. **Summary** Analyze the games in problem 1 using the table you made. Explain why some sums are more likely to occur than others and how this determines who wins more often.

Definition: A game is *fair* if each of the players is equally likely to win.

5. Is Game One fair? How about Game Two? Explain.

OUTCOMES AND EVENTS

Definition: We call one roll of the dice an *experiment*. Each of the different possibilities you listed in the table is called an *outcome* of the experiment.

6. When you roll a red die and a blue die, how many outcomes are possible?

7. If you flip a penny and a nickel, how many outcomes (heads and tails) are possible? Make a list.

8. If you roll a red, a blue, and a yellow die, how many outcomes are possible?

When an experiment is performed, we are usually interested in whether or not a particular *event* has occurred. An event consists of one or more outcomes.

In the two-dice experiment, an example of an event could be: *The sum of the dots is even.* This event was important in Game One of problem 1. In that game, 36 outcomes were possible. However, we were not interested in the individual outcomes, but only in which of the two events had occurred: an even sum or an odd sum.

9. In what events were we interested in Game Two of problem 1?

10. The outcome of a two-dice experiment is (3, 2). Which of the following events occurred?
    a. The difference is even.
    b. The product is even.
    c. One die shows a multiple of the other.
    d. The sum is a prime number.

The table you made in problem 2 was organized to show these *events*: the sum of the dots is 2, the sum of the dots is 3, etc. In that table, each column corresponds to one event. A table like the following one is another way to represent the two-dice experiment. It is organized around the *outcomes*. Each cell corresponds to one outcome.

Blue Die

Red Die	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
...	...	...	...	...	...	...

In the two-dice experiment, figure out how many outcomes make up each event in problems 11-14.

You can make the same kind of table to help answer problems 11-14. For example, to think about problem 11a, you would write the products in the cells.

Blue Die

Red Die	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	...	...	...	...

11. a. The product is even.
    b. The difference is even.
    c. One die shows a multiple of the other.

12. a. The sum is 2, 3, or 4.
    b. The sum is 9, 10, or 12.

13. a. a double
    b. not a double

14. a. The sum is prime.
    b. The product is prime.
    c. The difference is prime.

CREATE DICE GAMES

15. Name two events in the two-dice experiment that each consist of nine outcomes.

16. Name an event in the two-dice experiment that consists of:

a. 17 outcomes;

b. 19 outcomes.

17. Create a dice game that is fair. Write the rules. Then write an explanation of why the game is fair.

18. Create a dice game that appears to favor one player, but that actually favors the other. Or, make up a dice game that appears to be fair, but that actually favors one player. Write the rules and an explanation of the game.

DISCOVERY *THREE QUANTITIES, THREE CONSTRAINTS*

These problems were invented by algebra students. You may want to use colored slips of paper to solve them. In each one, there are three unknown quantities and three constraints. Try to find the three unknown quantities.

19. The red and yellow marbles add up to 5.
Blue and red add up to 7.
There are 8 yellows and blues altogether.

20. The blue and red add up to 9.
There are two times as many yellows as blues.
There are 15 marbles altogether.

21. Blue + 9 = Black
Blue times 3 = Red
Black + 1 = Red

22. $A + B = 11$
$A + C = 7$
$B + C = 6$
How many of each?

LESSON 11.6

What is Probability?

You will need:

two coins 

This lesson will introduce you to three interpretations of probability.

RELATIVE FREQUENCY

While waiting for his food at the Slow Food Café, Zoltan asked himself, "What is the probability of getting at least one head when tossing two coins?" He thought it might be 1/2, since there was an equal chance of getting heads or tails, or 1/3, since there were three possibilities (two heads, one head, no heads). He decided to find out by doing an experiment. Here are his notes on the first eight tosses (or *trials*).

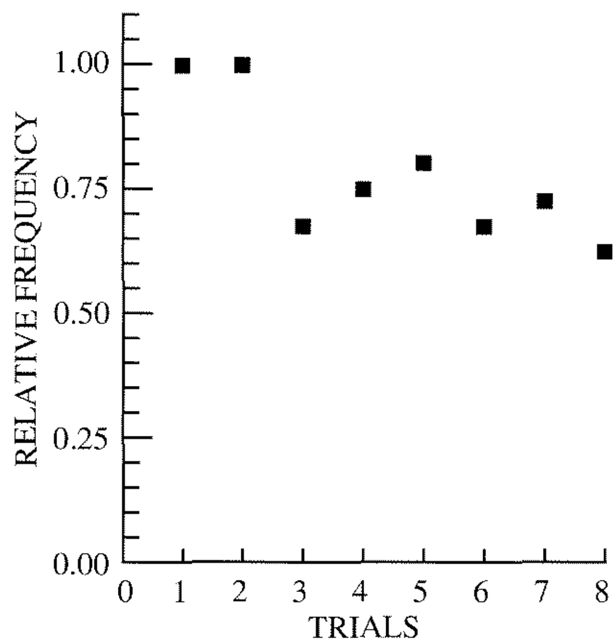
HH, HT, TT, HT, HH, TT, TH, TT...

He made a table of the results. A *success* is a toss where one or two heads appeared.

Definition: The *relative frequency* of the successes is the ratio of successes to trials.

Trials so far	Successes so far	Relative frequency
1	1	1/1 = 1.00
2	2	2/2 = 1.00
3	2	2/3 = 0.67
4	3	3/4 = 0.75
5	4	4/5 = 0.80
6	4	4/6 = 0.67
7	5	5/7 = 0.72
8	5	5/8 = 0.63

He graphed the results, with relative frequency on the y-axis, and trials on the x-axis.



1. Toss a pair of coins 30 times. Make a table like Zoltan's.
2. Make a graph like Zoltan's for the data in your table.
3. If you tossed the coins 100 times, what do you think your graph would look like? What if you tossed them 500 times? Explain.

First Definition: The *probability* of an event is often interpreted to mean the relative frequency with which that event occurs if the experiment is repeated many, many times.

Example: If you roll a die many times, you expect the relative frequency of threes to be approximately 1/6.

4. Explain why the relative frequency of an event is a number from 0 to 1.

EQUALLY LIKELY OUTCOMES

This definition is the most common interpretation of probability.

Second Definition: The *probability* of an event A is

$$P(A) = \frac{e}{t}$$

where:

e = the number of equally likely outcomes in the event.

t = the total number of equally likely outcomes possible.

Example: In the two-dice experiment, say that event D is the event that the sum is 8. Then

$$D = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

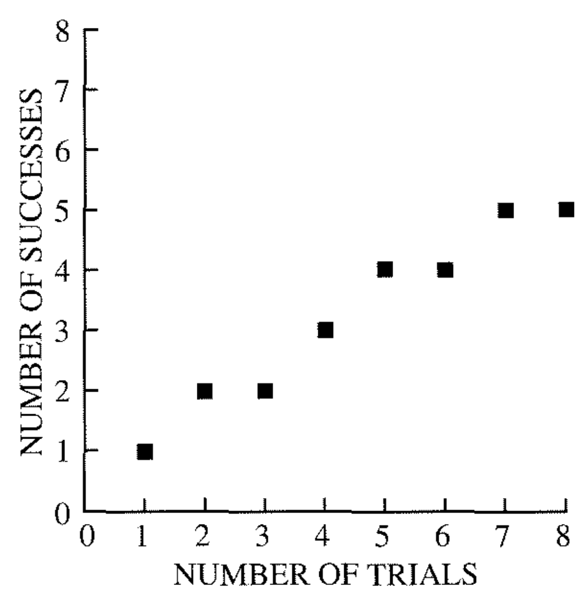
Since D consists of five equally likely outcomes, and the total number of equally likely outcomes is 36,

$$P(D) = \frac{5}{36}.$$

5. For the two-dice experiment, find an event having the following probabilities:
   a. $\frac{2}{36}$ b. $\frac{1}{12}$
6. For the two-dice experiment, find the probability of these events.
   a. The product is more than 25.
   b. The product is less than 50.
   c. The sum is 7 or 11.
7. Explain why any probability p will always satisfy the inequality $0 \le p \le 1$.
8. For the two-dice experiment, find an event having the following probabilities:
   a. 0 b. 1
9. List all the equally likely outcomes in Zoltan's two-coin experiment. (Hint: Think of the coins as a penny and a nickel. Make a table.)
10. What is the probability that there will be at least one head when tossing two coins? Explain.

THEORETICAL vs. OBSERVED PROBABILITY

Zoltan graphed his results another way. This time he put the number of successes on the y-axis and the number of trials on the x-axis.



11. Make a graph like Zoltan's for the data in the table you made in problem 1.

12. On your graph, draw lines having equations:

$$successes = trials$$
$$successes = 0.75 \cdot trials$$
$$successes = 0.67 \cdot trials$$
$$successes = 0.50 \cdot trials$$

13. What do rise and run each measure on this graph? What does slope represent?

On a graph like this, the *theoretical probability*, as predicted by the analysis of equally likely outcomes, can be represented as a line through the origin, having slope equal to the probability. The *observed probability* as seen in the experiment is represented by the slope of the line through the origin and the corresponding data point. Note that data points rarely land exactly on the theoretical line.

14. Which line that you drew in problem 12 represents the theoretical probability? Explain.

15. Add a line representing the theoretical probability to the graph you made in problem 2. Explain.

SUBJECTIVE PROBABILITY

A third interpretation of probability is *subjective probability*. This is the probability that a person assigns to an event based on his or her own knowledge, beliefs, or information about the event. Different people may assign different probabilities to the same events.

Example: Before Mark took his driving test, Karen said, "I think you've got about a 60% chance of passing."

What subjective probability would you assign for each of the following events? Explain your reasons.

16. It will be cloudy on a night with a full moon this month.

17. You will be assigned no math homework this Friday.

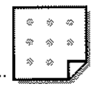
18. School will be cancelled next week due to bad weather.

19. Exactly half of the students in your math class next year will be boys.

LESSON 11.7

Random Walks

You will need:

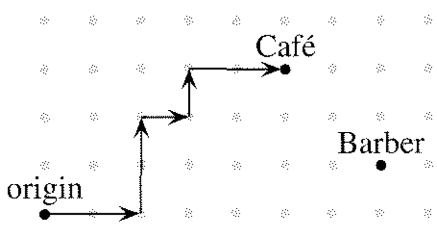
dot paper 

coins 
(pennies, nickels, dimes, quarters)

The Mad Probabilist takes a random walk on dot paper. Starting at the origin, he goes from lattice point to lattice point, flipping a coin each time to determine where to go next.

- *Heads* means to move east, increasing just the x-coordinate by 1.
- *Tails* means to move north, increasing just the y-coordinate by 1.

The map shows the path H, H, T, T, H, T, H, H.



1. **Exploration** Find another sequence of heads and tails that would get the Mad Probabilist from the origin to (5, 3), where the Slow Food Café is located. Compare your sequence with that of a classmate. How many ways are there to reach (5, 3)?

A FOUR-COIN EXPERIMENT

2. If you toss a penny, a nickel, a dime, and a quarter, which do you think is most likely to occur: 0 heads, 1 head, 2 heads, 3 heads, 4 heads? Or are they all equally likely? Explain your reasoning.

3. Use a penny, a nickel, a dime, and a quarter. Toss them and record the number of heads. Repeat this experiment 20 times.

If you toss a penny, a nickel, a dime, and a quarter, the event *three heads* consists of the following equally likely outcomes: HHHT, HHTH, HTHH, and THHH, depending on which coin comes up tails.

4. Find all possible equally likely outcomes when tossing four coins.

5. Count the outcomes for each of these events: 0 heads, 1 head, 2 heads, etc.

6. Are the results of your experiment in problem 3 consistent with your analysis in problems 4 and 5? Comment.

If you toss one coin, there are two equally likely possible outcomes, H and T. In Lesson 6 you studied the tossing of two coins, (HH, HT, TH, TT), and in problems 5-6 the tossing of four coins.

7. Figure out how many equally likely outcomes are possible if you toss
   a. three coins; b. five coins.

8. **Generalization** How many equally likely outcomes are possible if you toss n coins? Explain.

Tossing the same coin repeatedly works in a similar way. For example, one possible string of eight tosses is: TTHTHTTH, just as one possible outcome of tossing eight coins is TTHTHTTH.

9. If you toss one coin eight times, how many possible outcomes are there? How about n times?

THE MAD PROBABILIST

10. How many moves does it take the Mad Probabilist to get to (5, 3)?

11. Generalization How many moves does it take him to get to (p, q)? Explain.

12. a. Where might he be after six moves?
b. Make a list of the points he could get to in seven moves.

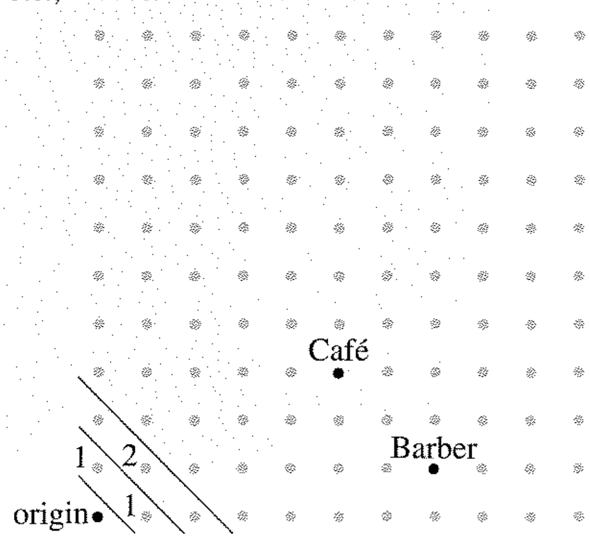
13. How would you describe the set of points you listed in problem 12b? (How many points does it consist of? What equation relates their coordinates?) Explain.

14. Generalization Describe the set of points he could reach in n moves. Explain.

15. Which is greater, the number of possible points he could end up on after eight tosses of a coin, or the number of possible strings of eight tosses? Explain.

MAKING A MAP

The Mad Probabilist wants to calculate the *probability* of getting to a lattice point like (5, 3). He decides to make a map on a piece of dot paper. He draws diagonal lines to separate the points he may reach in one, two, three, etc., moves.



Then he writes how many ways there are to reach each point on the map. For example, there is only one way to get to (1, 0): a toss of H. There is only one way to get to (0, 1): T. There is only one way to get to (2, 0): HH. There are two ways to get to (1, 1): HT or TH.

As he makes his map, he finds it helpful to ask himself for each point, "Where could I have come from to get here?"

16. Continue the Mad Probabilist's map, until you get to (5, 3).

The Mad Probabilist reasons, "At the end of eight moves, I will be at one of these points, one of which is the Slow Food Café." He marks the points on his map. "The outcomes are eight-move paths; the event is those paths that end up at (5, 3). To find out the probability of this event, I need a numerator and a denominator." He writes:

$$P(5, 3) = \frac{\text{\# of paths that get to (5, 3)}}{\text{\# of 8-move paths}}$$

17. What is $P(5, 3)$? In other words, what is the probability the Mad Probabilist's random walk will end up at the Slow Food Café?

18. What is the probability it will end up at (7, 1), where the barbershop is? Explain.

19. Summary Explain how you can find the probability of getting to any lattice point in the first quadrant.

DISCOVERY ***PASCAL PATTERNS***

This is one of the most important arrays of numbers in mathematics. It is called Pascal's triangle.

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

20. **Exploration** Study this triangle, looking for patterns. Explain any patterns that you find.

21. Find a pattern that will enable you to write the next row in the triangle.

22. Find the pattern in the third column.

23. Find the pattern in the sums of the rows.

24. Find the pattern in the sums of the upward diagonals.

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

25. **Report** Write an illustrated report about the patterns you found in Pascal's triangle. Include a section on the relationship between Pascal's triangle and coin-tossing experiments.

LESSON 11.8

Unit Conversion

You will need:

graph paper

1. Find the missing numerator and denominator in each equation. (You should be able to solve most of these without multiplying or dividing.) Compare your answers with other students' answers.

   a. $\frac{2}{3} \cdot \frac{5}{7} \cdot \frac{\triangle}{\square} = \frac{2}{3}$

   b. $\frac{2}{3} \cdot \frac{\triangle}{7} \cdot \frac{5}{\square} = \frac{2}{3}$

   c. $\frac{7}{12} \cdot \frac{14}{11} \cdot \frac{12}{5} \cdot \frac{\triangle}{\square} = \frac{7}{11}$

   d. $\frac{1}{3} \cdot \frac{5}{x} \cdot \frac{2}{5} \cdot \frac{x}{2} \cdot \frac{\triangle}{\square} = \frac{8}{3}$

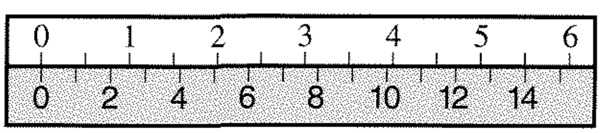
   e. $\frac{8}{7} \cdot \frac{3}{x} \cdot \frac{7}{6} \cdot \frac{y}{\square} \cdot = \frac{y}{x}$

   f. $\frac{a}{b} \cdot \frac{\triangle}{a} \cdot \frac{b}{\square} = \frac{x}{y}$

TWO RULERS

Alice had a new ruler. Oliver suggested she measure it with another ruler, as in this figure.

Alice's ruler



Oliver's ruler

Oliver and Alice had to write about functions for algebra. They decided to use the rulers as a way to get tables of x- and y-values. Here are the tables they got from the ruler setup.

Alice

x	y
1	2.5
2	5.1
3	7.6

Oliver

x	y
1	0.4
2	0.8
3	1.2

2. Describe the pattern for the numbers in each table.

3. What do you think the units of each ruler are?

4. Write a function of the type $y =$ *an expression in terms of* x for each table. (Because of measurement error, this may have to be an approximation.)

5. If you were to graph these functions, explain why the graph would
   a. be a line;
   b. pass through the origin.

6. According to an almanac, 1 inch = 2.54 centimeters, exactly. Using that information, what is the exact length of a centimeter, in inches?

When converting inches to centimeters, we multiply by 2.54. When converting centimeters to inches, we multiply by 1/2.54. As you can see from the equations, this conversion of units involves direct variation.

Definition: In the case of unit conversion, the proportionality constant (the number you multiply by) is called the *conversion factor.*

Conversion factors have units. For example, the conversion factor from inches to centimeters is 2.54 *cm/in*.

7. What is the conversion factor from centimeters to inches? (Include its unit.) Explain.

MULTIPLYING BY ONE

When converting a quantity from one unit to another, the way the quantity is measured is changed, not the amount of it. We can think of the conversion factor as having the value 1.

> **Example:** Two miles are how many feet?
>
> $$2 \text{ miles} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 10{,}560 \text{ feet}$$

The conversion factor is 5280/1 and its units are feet/mile. Since 5280 feet = 1 mile, the numerator equals the denominator in the fraction, so we can think of this conversion as multiplying by a form of 1. To make the units work out, we multiplied by feet and divided by miles.

In problems 8-10, when writing a conversion factor, include its unit.

8. a. What is the conversion factor used to convert feet to miles?

b. Mount Everest, the world's tallest peak, is 29,028 feet high. How many miles is that?

9.

a. What is the conversion factor used to convert seconds to minutes?

b. What is the conversion factor used to convert minutes to seconds?

c. How are the answers to (a) and (b) related? Explain.

10. Convert 1000 inches to:

a. feet; b. miles;

c. meters; d. kilometers.

TWO-STEP CONVERSIONS

In science, speeds are sometimes given in feet per second. To convert feet per second to miles per hour, there are two steps:

- Convert feet to miles.
- Convert seconds to hours.

The steps can be combined:

$$\frac{\text{feet}}{\text{second}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} = \frac{\text{miles}}{\text{hour}}$$

We chose the conversion factors in order to divide by feet and multiply by seconds so that those units did not appear in the final answer.

11. Convert the speed of sound in cold water (4938 feet per second) to miles per hour. Show your calculations.

12. To convert feet per second to miles per hour, what single number could you multiply by? Explain how you obtained this conversion factor.

13. Find the conversion factor between each of these common measures of speed. Show all your work. Summarize your results in a table like this one. Give approximations to the nearest thousandth. (Note: m/sec means meters per second.)

From: \ To:	mi/hr	km/hr	m/sec	ft/sec
mi/hr	—	—	—	—
km/hr	—	—	—	—
m/sec	—	—	—	—
ft/sec	—	—	—	—

14. In your table, find pairs of numbers that are reciprocals of each other. Explain why they should be reciprocals.

15. Use your table to convert
   a. the speed of light (299,792,500 m/sec) into miles per hour;
   b. the speed of sound in cold air (1,088 ft/sec) into miles per hour.

16. A fast runner can run a mile in four minutes. How fast is that in miles per hour?

17. **Project** Find out how fast students in your class walk, skip, run, move backwards, etc., by timing how long it takes them to cover a measured distance. Convert the speeds to miles per hour.

18. **Project** Find out how fast cars drive on a nearby street or road, by timing how long it takes them to cover a measured distance. Convert the speeds to miles per hour.

REVIEW *SOLVING SYSTEMS*

Solve each system. Check first to see if you can tell that the system has no solution or an infinite number of solutions.

19. $\begin{cases} 2x + 6 = 3y \\ 4y = 12 - 3x \end{cases}$

20. $\begin{cases} -m - b = 25 \\ -m + b = 13 \end{cases}$

21. $\begin{cases} 2r + 2s = 60 \\ r - 2s = 5 \end{cases}$

22. $\begin{cases} 2m + n = -1 \\ m + 3n = -18 \end{cases}$

23. $\begin{cases} r - s = 1 \\ r + 3s = -11 \end{cases}$

24. $\begin{cases} \frac{2}{3}x + \frac{2}{5}y = 4 \\ x - 2y = 5 \end{cases}$

25. $\begin{cases} y = \frac{3}{7}(x - 8) \\ y - 4 = \frac{3}{7}(x + 6) \end{cases}$

THINKING WRITING

11.B Calibrating a Speedometer

You can check the accuracy of a car's speedometer by using a stopwatch and the mile markers on a highway. The driver should maintain a steady speed while a passenger uses a stopwatch to time the travel time between mile markers. This travel time tells you the number of seconds it takes you to go one mile, which you can convert to miles per hour.

1. Convert 0.123 hours to minutes and seconds.

2. Convert 4.567 hours to hours, minutes, and seconds.

3. A car is traveling at 55 miles per hour.

a. What fraction of an hour does it take to go one mile?

b. How many minutes and seconds does it take to go one mile?

c. How many seconds does it take to go one mile?

4. How would you convert

a. miles per hour to miles per second?

b. miles per second to miles per hour?

c. miles per second to seconds to go one mile?

d. seconds to go one mile to miles per second?

5. If it takes you 65 seconds to go one mile, how many miles per hour are you going? Explain how you figured this out, showing calculations.

6. Describe a general strategy for converting seconds per mile to miles per hour.

7. Make a table like this one to help people check their speedometers.

Seconds between mile markers	Speed in mi/hr
...	...

8. a. Graph the ordered pairs in the table you made.

b. Let y stand for the speed in mi/hr, and x stand for the number of seconds between markers. Write an equation relating x and y.

Say that the person in charge of timing can be off by one second in starting the stopwatch, and one second in stopping it.

9. What is the maximum error in using the table, resulting from the inaccuracy in timing?

10. If, instead of measuring the time to go one mile, you measure the time to go four miles and use the average one-mile time, what is the maximum error?

11. Report Write an explanation for the general public of how to check the accuracy of a speedometer. Include your table, some illustrations, and an explanation of what to do to get an exact answer between values given in the table.

Essential Ideas

SUMS

1. Find each sum.

a. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$

b. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$

c. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5$

d. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ... + \left(\frac{1}{2}\right)^n$

2. This sum goes on for ever. (We call it an *infinite series*.) Use the pattern you found in problem 1 to estimate the sum of this infinite series.

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ...$$

3. Estimate the sums of these infinite series.

a. $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + ...$

b. $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + ...$

c. $\frac{1}{k} + \left(\frac{1}{k}\right)^2 + \left(\frac{1}{k}\right)^3 + ...$

(Assume that k is a positive integer.)

GEOMETRIC SEQUENCES

4. Some of the following sequences are geometric; find their common ratio. Some are arithmetic; find their common difference.

a. $2/3, (2/3)^2, (2/3)^3, (2/3)^4, ...$

b. 1/3, 4/3, 7/3, 10/3, ...

c. 10, 10/8, 10/64, 10/512, ...

d. 10, 80, 640, 5120, ...

e. 1/3, 8/3, 64/3, 512/3, ...

5. Find the sum of the first 50 terms for the sequences in problems 4a and e.

6. Two of the sequences in problem 4 are such that if you add the entire infinite sequence, the sum converges to a finite number.

a. Explain how you can tell which sequences they are.

b. Find the sum they each converge to.

INHERITANCE

The brothers Able and Earl inherited from their father an acre of land, which they divided equally. Each brother willed his land to his family. Able's family was large, and Earl's was small. Able's family needed more land, so they bought 40% of the land belonging to Earl's family. In the next generation, Able's family again bought 40% of Earl's family land. This continued for several generations.

7. Copy and and extend this table to show the amount of land owned by each family up to the eighth generation.

Generation	Able's land	Earl's land
1	0.5	0.5
2	0.7	0.3

8. Study the data. At this rate, will Able's family ever own the whole acre? Explain.

DECIMALS AND FRACTIONS

9. Write as a fraction.

a. $0.\overline{21}$

b. $0.3\overline{21}$

c. $0.\overline{321}$

10. Find whole numbers p and q such that:

a. $0.45 < p/q < 0.46$

b. $0.\overline{4} < p/q < 0.45$

PRIME FACTORIZATION

11. Explain why the square of an even number must be a multiple of four.

12. Explain why the square of an odd number must be odd.

13. Explain why the double of an odd number is an even number, but not a multiple of four.

LATTICE POINTS

Imagine that you are standing at the origin, and that you cannot see lattice points that are hidden behind other lattice points. For example, you cannot see (2, 2) because (1, 1) is in the way. Let us call (1, 1) *visible* and (2, 2) *hidden*.

14. List three visible lattice points and three hidden ones. Explain.

15. By looking at its coordinates, how can you tell whether a lattice point is visible?

16. Give the equation of a line that includes no lattice points except the origin.

17. 💡 Give the equation of a line that includes no lattice points at all.

18. 💡 Which line on an 11-by-11 geoboard contains the greatest number of visible lattice points?

GAMES AND PROBABILITY

19. If you choose a letter at random from the alphabet, what's the probability that it's a vowel?

20. If you choose a month at random, what's the probability that its name

a. begins with J?

b. contains an R?

21. Assume that you draw one card from an ordinary deck of 52 playing cards. What's the probability that you draw

a. a 7?

b. a heart?

c. a 7 or a heart?

d. a 7 of hearts?

22. Which game, if either, is fair? Explain.

a. Roll a pair of dice and multiply the numbers on the uppermost faces. If the product is 18 or greater, Player A wins. If the product is less than 18, Player B wins.

b. Toss three coins. If the number of heads is even, Player A wins. If it is odd, Player B wins.

c. 💡 Repeat part (b) for six coins.

UNIT CONVERSION

23. Given that 1 pound is approximately 454 grams, 1 kilogram is approximately how many pounds?

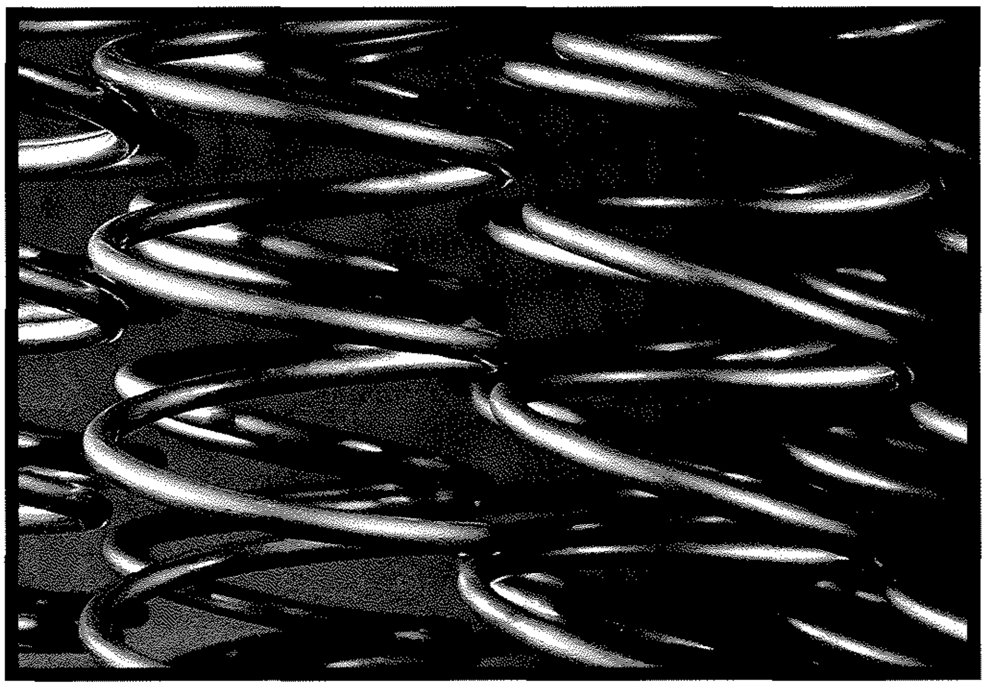
24. Find conversion factors for converting the following measurements. (Note: Even though these problems look different, you can use the technique you learned in Lesson 8. Remember that in.^2 means in. · in.)

a. in.^2 to ft^2 b. ft^2 to in.^2

c. in.^3 to cm^3 d. cm^3 to in.^3

25. The density of water is approximately 1 gram/cm^3. What is it in pounds/ft^3?

CHAPTER 12



The spiral coils of car springs

Coming in this chapter:

Exploration By measuring people's feet and asking for their shoe size, find a formula relating foot length, in inches, to shoe size,

a. for men;
b. for women.