Square–Sum Pair Partitions

Gordon Hamilton, Kiran S. Kedlaya, and Henri Picciotto

  

Gordon Hamilton (gord@mathpickle.com) has a Ph.D. in mathematical biology from the University of Calgary and is now a puzzle and board game designer and the director of MathPickle.com. MathPickle's primary objective is to get 13 curricular, unsolved problems into classrooms worldwide—one for each grade K–12. He also enjoys spending time with his two children. **Kiran S. Kedlaya** (kedlaya@ucsd.edu) is a professor of mathematics at the University of California, San Diego, having previously been a faculty member at the Massachusetts Institute of Technology. He is a long-time contributor to the USA Mathematical Olympiad and co-authored the third MAA compilation of the Putnam competition. He is an avid travel photographer and placed fourth at the 2015 American Crossword Puzzle Tournament. **Henri Picciotto** (henri@mathedpage.org) is a math education consultant and author; he shares ideas on www.MathEducationPage.org. He received his B.A. and M.A. in mathematics from the University of California, Berkeley. Picciotto retired from the classroom after 42 years of teaching at every level from counting to calculus. He is interested in puzzles, manipulatives, and technological teaching tools. His cryptic crosswords appear in *The Nation* every week.

Here is a classic problem aimed at students in grades 5 to 7.

Exercise 1. Arrange the whole numbers from 1 to 18 into nine pairs so that the sum of the numbers in each pair is a perfect square.

We will consider several generalizations of the problem in this article, many of which are well-suited for use in teacher preparation and in undergraduate discrete mathematics or introduction to proof courses. The following features of the problem make it a good choice.

1. The prerequisites are minimal (recognizing perfect squares, being able to add pairs); thus, it is accessible to a wide range of students.
2. The answer is not obvious.
3. Students' insights increase as they explore the problem.
4. Partial solutions are possible (e.g., one can find eight pairs that satisfy the condition).
5. The problem can be generalized.

The fourth point suggests a rewording of the problem.

Exercise 1'. Arrange the whole numbers from 1 to 18 into nine pairs so that the sum of the numbers in *as many pairs as possible* is a perfect square.

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This is perhaps a better place to start because students can immediately find solutions, and a healthy competition develops around finding solutions with more and more pairs.

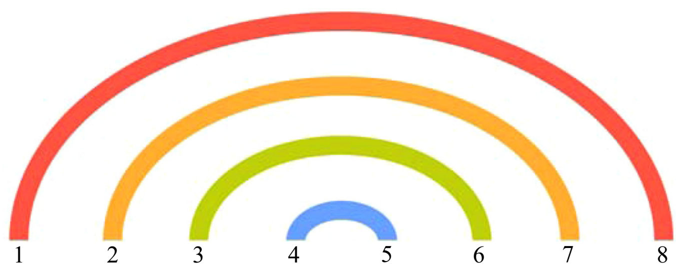
Solving this problem provides an arena for comparing student approaches. For example, is it better to start with small pairs such as {1, 3} or large ones such as {17, 8}? The latter is in fact more efficient since it presents fewer choices early on. An optimal strategy is suggested in [**6**, p. 191].

When the nine-pair solution has been found, i.e., when {1, 2, . . . , 18} has been partitioned into square–sum pairs, the class can launch into the most natural generalization.

Exercise 2. For what numbers is this possible?

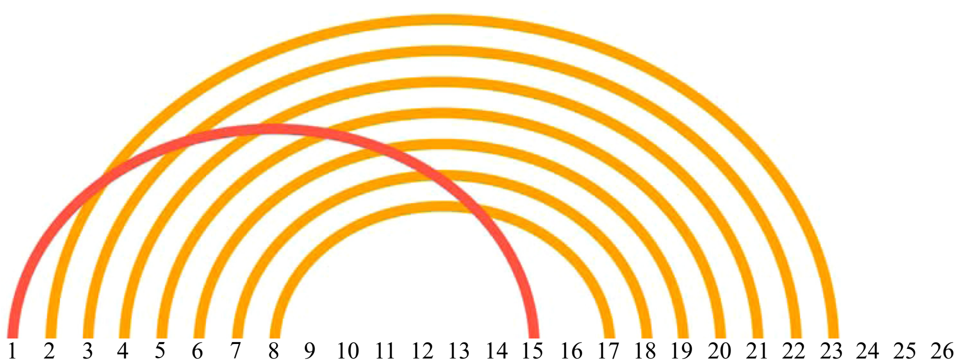
In an initial trial-and-error exploration, students will find that this partition into square–sum pairs can be done, e.g., for 14 and 18 but not for 12 or 20. A systematic exploration, perhaps conducted by small groups of students, shows that most even numbers between 2 and 22 do not yield a solution. However, students may be surprised to learn that, in fact, a square-sum pairs partition is possible for *all* even numbers greater than 22. This is a strong argument for moving beyond trial and error, as such experimentation becomes increasingly difficult and time consuming when numbers get larger. Using a computer program becomes a useful complement to manual exploration. One such program by the third author can be found under "Links" at [**3**, A253472].

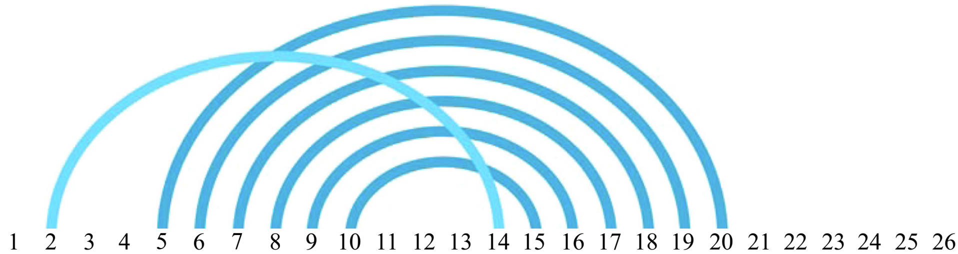
The set {1, 2, . . . , 8} offers a particularly straightforward solution: {1, 8}, {2, 7}, {3, 6}, {4, 5}. This suggests an elegant visual "rainbow" representation.



We will call this a rainbow pairing. The technique can be used to create problems for precollege explorations in an accessible, puzzle-like format, such as the following.

Exercise 3. Complete these two square–sum pair partitions of the set {1, 2, . . . , 26}.





See [**1**] for more examples and an engaging story involving leprechauns!

Rainbow pairing is a helpful tool in the following proof by induction that gives a complete answer to Exercise 2.

Theorem. *For n a positive integer, the set $\{1, 2, \ldots, 2n\}$ admits a partition into square–sum pairs except when $n \in \{1, 2, 3, 5, 6, 10, 11\}$.*

Proof. We will proceed by strong induction on n, treating all of the cases for $n \leq 30$ as base cases. These are most easily handled by a computer search, as suggested earlier; while many can be treated by hand, a few (especially $n = 29$) are quite intricate.

Our strategy will be to construct a rainbow pairing on $\{2m + 1, \ldots, 2n\}$ for some m, pairing $2m + 1$ with $2n$, $2m + 2$ with $2n - 1$, and so on. Then we can invoke the induction hypothesis with n replaced by m to finish. For this to work, we require the following conditions.

- $m < n$, for this construction to make sense.
- $m \geq 12$, to avoid the known exceptional cases.
- $2m + 2n + 1$ is a perfect square.

Since this square is odd, it must have the form $(2k + 1)^2$ for some nonnegative integer k, so the third condition can be written as $m + n = 2k^2 + 2k$.

For each k, we see that we can complete the induction in the cases $n = k^2 + k + 1$ up to $n = 2k^2 + 2k - 12$ (using the bound on m). We want this to overlap with the analogous list for $k + 1$, i.e., we need $2k^2 + 2k - 12 \geq (k + 1)^2 + k + 1$. It is straightforward to show that this inequality holds for $k \geq 5$. Since the list for $k = 5$ starts at $n = 31$, the lists together cover all of the integers $31, 32, \ldots$ without any gaps. This means that for each $n > 30$, we indeed have a construction that reduces the problem to an earlier case. ■

A note on the proof: If one replaces the condition $m \geq 12$ with the more precise $m \notin \{1, 2, 3, 5, 6, 10, 11\}$, then with care the number of base cases can be reduced from 30 to 17, eliminating many of the higher values, although $n = 29$ and $n = 30$ remain. The reduction comes primarily from considering $k = 4$.

Exercise 4. There is a similar theorem for the “odd” version of the problem, where the initial number set is $\{0, 1, \ldots, 2n - 1\}$. State and prove the theorem.

See [**2**, p. 77] for one approach to Exercise 4.

In Exercise 4, we have an input number set different from the previous problems. Other related changes are possible, such as in the following two exercises.

Exercise 5. For an integer p, is there a similar theorem for $\{p, \ldots, p + 2n - 1\}$?

From this point of view, our theorem is the case $p = 1$ and Exercise 4 is the case $p = 0$; see [**1**].

Exercise 6. Even more generally, is there a similar theorem for a general $2n$-term arithmetic progression $\{p, p+d, p+2d, \ldots, p+(2n-1)d\}$?

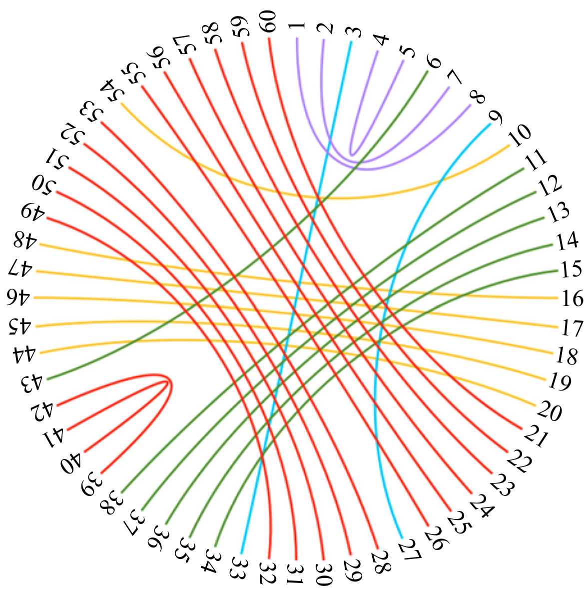
Changing targets

Using square numbers as targets produces some beautiful puzzles, but the number of solutions explodes. Table 1 shows the number of solutions for $\{1, 2, \ldots, 2n\}$ from [**3**, A252897].

Table 1. Number of square–sum pair partitions of $\{1, 2, \ldots, 2n\}$ for $1 \leq n \leq 30$.

1	2	3	4	5	6	7	8	9	10
0	0	0	1	0	0	1	1	1	0
11	12	13	14	15	16	17	18	19	20
0	1	6	18	12	36	156	295	429	755
21	22	23	24	25	26	27	28	29	30
2603	7122	19232	32818	54363	172374	384053	933748	1639656	4366714

Here is one of the 4,366,714 solutions for $\{1, 2, \ldots, 60\}$.



One way to constrain the problem is to look for solutions that minimize the number of target squares. (Looking at the coloring/position of the arcs, one can see that the solution above uses five squares for sums.)

Squares are not the only possible targets for problems of this type. Below are some interesting variations.

Exercise 7. Explore using powers of 2 as targets.

Exercise 8. Explore using numbers that are one less than a power of 2 as targets.

Exercise 9. Explore using prime numbers as targets. (This is solved in [**2**, p. 78].)

Exercise 10. Explore using Fibonacci numbers as targets.

In an unpublished 2003 paper, "Fibonacci plays billiards," Elwyn Berlekamp and Richard Guy show that, unlike in the case of squares, the number of Fibonacci–sum pair partitions does not grow quickly.

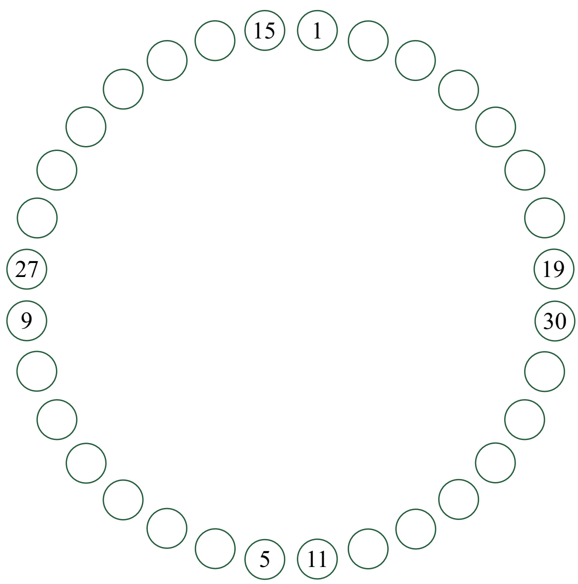
Among the many more possibilities, one could vary both the input set (as in Exercises 4–6 for square–sum pairs) and the target numbers (Exercises 7–10). We hope that the reader will explore some of them and find ones that provide satisfying puzzles or interesting proofs. (For example, when the targets are given by the values of a polynomial, one can imitate our earlier proof to show that pairings always exist once n is sufficiently large.)

Finally, here are two puzzles from [**4**] that are closely related to Exercise 1. Bernardo Recamán suggests that the numbers $\{1, 2, \ldots, n\}$ be ordered in a row so that adjacent numbers sum to a square.

Exercise 11. Lay out the numbers from 1 to 15 so that adjacent numbers sum to a square. (This is the smallest value for which the puzzle can be solved.)

Joe Kisenwether asks a similar question but for a necklace of numbers—again adjacent numbers must sum to a square.

Exercise 12. Four pairs of neighbors are given below. Fill in the remaining beads so that all adjacent numbers sum to a square and all the numbers from 1 to 32 have been used once. (This is the smallest value for which the puzzle can be solved.)



The square–sum pair partition problem and its variations combine both access and challenge in one easy-to-present package. Undergraduates should find it engaging. If some of them become teachers, they will be able to share it with their precollege students in math classes and math clubs.

Acknowledgment. Thanks to Joshua Zucker for providing the references [**2, 6**].

Summary. We present a middle school problem and generalize it in several ways, including many new variations for readers to explore. The problems should be attractive to students, as they have few prerequisites and lend themselves to beautiful visual representations.

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