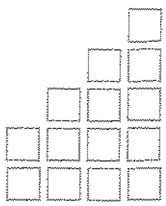
LESSON 5.9

Staircase Sums

ONE STEP AT A TIME

Here is an example of a kind of arrangement that we'll call a *staircase.* It has 4 steps and the first step is of height 2.



Definition: For this lesson, we will define a *staircase* as a sequence of stacks of tiles in which each stack is one tile higher than the previous stack. *There must be two or more steps in the staircase,* and the first step can be of any height.

1. How many tiles would you need to build each of these staircases?
   a. First step: 7 Number of steps: 8
   b. First step: 8 Number of steps: 7
   c. First step: 6 Number of steps: 9

2. There are two different nine-tile staircases: $2 + 3 + 4$ and $4 + 5$.
   a. Find three different 15-tile staircases.
   b. Find four different 105-tile staircases.

3. **Exploration** Find every possible staircase with each number of tiles from 2 to 34. Hints:
   - Work with other students.
   - Keep organized records of your work.
   - It is not necessary to draw the staircases.
   - Look for strategies: What numbers can be made into two-step staircases? Three-step?
   - Look for patterns: What numbers are easiest? What numbers are impossible?

4. The number 10 can be written as the sum of four consecutive numbers.
   a. What are these four numbers?
   b. If negative integers and zero are allowed, can the number 10 be written as the sum of consecutive numbers in any other way? If so, show how.

5. Show how the number 4 can be written as a sum of consecutive integers if negative numbers and zero can be used.

6. **Generalization** What is the maximum number of consecutive integers that can be used to write the number 17 as a sum? What is the maximum number of consecutive integers that can be used to write the number N as a sum? (Assume N is an integer.) Explain your answer, giving examples.

SUMS FROM RECTANGLES

7. a. On graph paper, sketch the staircase illustrated at the beginning of the lesson. Then make a rectangle by sketching a copy of the staircase *upside down* on the first staircase. (You can also do this by building the staircases with tiles.)
   b. What are the length, width, and area of the rectangle?

8. Imagine a staircase having 100 steps, and a first step of height 17.
   a. It would be half of what rectangle? (Give the length and width.)
   b. How many tiles would you need to build the staircase? Explain how you know.

9. Show how you could find the sum of:
   a. the integers from 5 to 55, inclusive;
   b. the integers from 0 to 100, inclusive.

GAUSS'S METHOD

Math teachers like to tell a story about Carl Friedrich Gauss. One day in elementary school he was punished by his teacher who asked him to add up all the whole numbers from 0 to 100. Carl immediately gave the answer, to his teacher's amazement. He grew up to be one of the greatest mathematicians of all time.

Gauss's method was to imagine all the numbers from 0 to 100 written from left to right, and directly beneath that, all the numbers written from right to left. It would look like this:

0	1	2	3	4	5	6	7	8	9...
100	99	98	97	96	95	94	93	92	91...

He mentally added each column, getting 100 each time. He multiplied 100 by the number of columns, and did one more thing to get the correct answer.

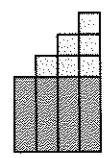
10. Finish Gauss's calculation. Be sure to use the correct number of columns, and to carry out the final step. Did you get the same answer as in problem 9b?

11. What would happen if the numbers to be added started at 1 instead of 0? Obviously, the sum should be the same. Would Gauss's method still give the same answer? Explain.

12. **Summary** You now know two methods for calculating staircase sums: one involves making a rectangle; the other is Gauss's method. Both methods work well, but it is easy to make mistakes when using them. Write a paragraph explaining how you would use each method to calculate the sum, $5 + 6 + 7 + ... + 89$. Use sketches as part of your explanation. Both methods should give the same answer.

VARIABLE STAIRCASES

You can build staircases with the Lab Gear. This diagram shows $(x) + (x + 1) + (x + 2) + (x + 3)$.



13. In terms of x, what is the sum of $(x) + (x + 1) + (x + 2) + (x + 3)$?

14. Find the sum of $(x) + (x + 1) + (x + 2) + (x + 3)$ if:
    a. $x = 4$;   b. $x = 99$.

15. Find each sum. Explain how you got your answer.
    a. $(x) + (x + 1) + (x + 2) + ... + (x + 26)$
    b. $(x + 1) + (x + 5) + ... + (x + 84)$

16. **Generalization** What is the sum of each staircase?
    a. $1 + 2 + 3 + ... + n$
    b. $(x) + (x + 1) + ... + (x + n)$
    c. $(x + 1) + ... + (x + n)$