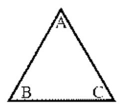
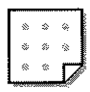
LESSON 5.12

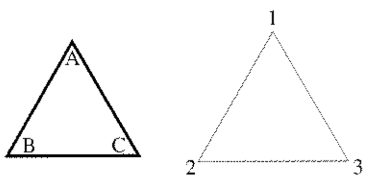
Smooth Moves

You will need:

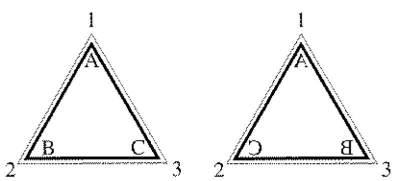
a cardboard equilateral triangle 

dot paper 

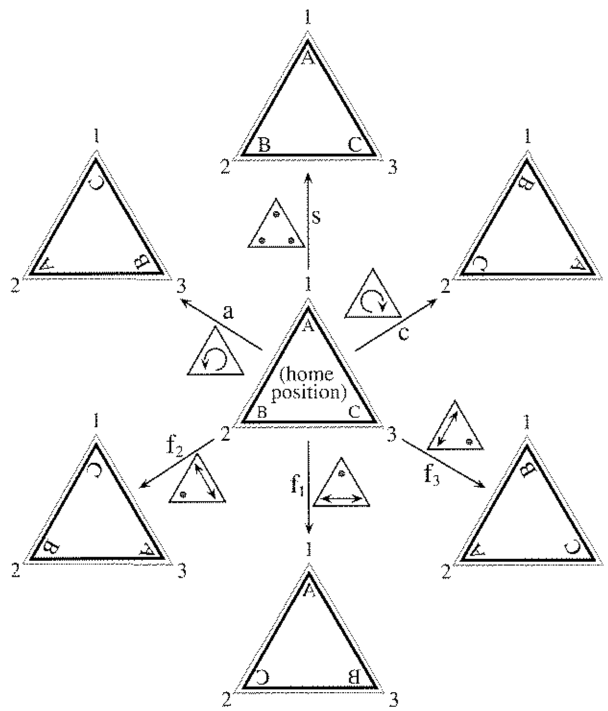
FLIPS AND TURNS



1. a. Write the letters A, B, and C on your triangle, near the vertices. Make sure the same letter appears on both sides of the cardboard at each vertex.
   b. Outline the triangle on a piece of paper, and write the numbers 1, 2, and 3 outside the outline, as in the figure.



There are several different ways you can place the triangle on its outline. The two ways shown in the figure can be written ABC and ACB. ABC is called the *home position.*



∘ means *this corner does not move.*

As you can see on the figure, you can get from the home position to each other position by using one of the following moves.

Turns:

- the clockwise turn (abbreviation: *c*)
- the counterclockwise turn (abbreviation: *a* — short for *anticlockwise*)

To do the turns (also called *rotations*), you do not lift the triangle off the page. You turn it until the triangle fits into the outline again.

Flips:

There are three flips. To do a flip you keep one corner in place and have the other two switch positions. For example, for flip 2 (f_2), you keep corner 2 fixed, and corners 1 and 3 switch positions. (Flips are also called *reflections.*)

Stay:

the *move* that does not move (abbreviation: s)

2. Which corner stays fixed and which changes position
a. for flip 3 (f_3)?
b. for flip 1 (f_1)?

Practice the turns and flips, making sure you know what each one does. In this lesson, you will have to execute turns and flips in succession, without going back to the home position in between.

Example: Do f_1, then a. (Such a sequence is simply written f_1a.) If you start at the home position, and do these moves in order, you will end up in the position BAC. (Try it.) But since you could have ended up there *in one move* (f_3), you can write: $f_1a = f_3$.

3. Find out whether $a f_1 = f_3$

4. Simplify. That is, give the one move that has the same result as the given sequence of moves.
a. $a\,a$ b. $f_1 f_3$ c. $f_3 f_1$
d. $s f_2$ e. $a\,c$ f. $c\,a$

5. Simplify.
a. $f_1 f_2 f_3$ b. $a f_1\, a f_2\, a f_3$
c. $f_1\, a f_2\, a f_3\, a$ d. $c f_1\, c f_2\, c f_3$

6. Figure out a way to write each of the six moves in terms of only f_1 and c.

7. Fill in the blanks.
a. a ___ $= f_1$ b. ___ $a = f_1$
c. f_1 ___ $= f_2$ d. ___ $f_1 = c$

THE ALGEBRA OF MOVES

Executing moves in order is an *operation* on triangle moves, just as multiplication is an operation on numbers. The set of six moves, together with this operation, is called the *symmetry group* for the triangle.

8. Make a *multiplication table* for triangle moves. That is, figure out the one move that has the same result as doing the two given moves. Describe any interesting patterns you find in the finished table.

Then...

First...	s	a	c	f_1	f_2	f_3
s	—	—	—	—	—	—
a	—	—	—	f_3	—	—
c	—	—	—	—	—	—
f_1	—	—	—	—	—	—
f_2	—	—	—	—	—	—
f_3	—	—	—	—	—	—

9. For each of the six moves, what move *undoes* it?

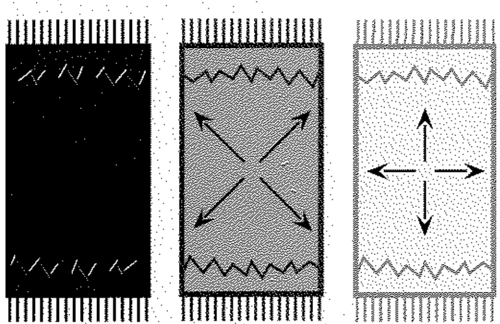
Executing one move (or sequence) repeatedly can be written with *power notation.* For example, f_2^7 means *execute f_2 seven times.*

10. Simplify.
a. a^{999} b. c^{1000}
c. f_2^{1000} d. $(af_2)^{1001}$

11. **Project** What flips and turns are possible for another figure, like a rectangle or a square? Write a report on the symmetry group for that figure.

DISCOVERY *MAGIC CARPETS*

Imagine that you can travel from dot to dot on dot paper, using magic carpets such as the ones illustrated in this figure. Carpets cost only \$1, plus \$1000 per arrow.



De Luxe \$8001 | Model X \$4001 | Carpet Plus \$4001

Magic carpets move in carpet steps. Each step takes the carpet and its riders to the next dot in the direction of one of the carpet's arrows. Each step takes one second. Carpets do not turn, so that the Carpet Plus cannot move diagonally, and the Model X cannot move horizontally or vertically.

Say you want to go from the origin to (6, 4). Here is a way to get there on each of the three carpets shown.

De Luxe: $\rightarrow\rightarrow\nearrow\nearrow\nearrow\nearrow$

Model X: $\nearrow\nearrow\nearrow\nearrow\nearrow\searrow$

Carpet Plus: $\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\uparrow\uparrow\uparrow\uparrow$

12. Find another way to get to (6, 4) on each of the three carpets.

13. Compare the advantages and shortcomings of the three carpets. Keep in mind cost, speed, and ability to reach any dot.

14. Project

a. Experiment with various \$3001 carpets. What are the advantages and shortcomings of each design? Again, keep in mind cost, speed, and ability to reach any dot. Give a full explanation of your findings.

b. Repeat part (a) for \$5001 carpets.

Using the directions North, East, West, and South instead of the arrows, the three examples given above could be written:

De Luxe — E E (NE) (NE) (NE) (NE)

Model X — (NE) (NE) (NE) (NE) (NE) (SE)

Carpet Plus — E E E E E E N N N N,

or even:

De Luxe — $E^2\,(NE)^4$

Model X — $(NE)^5\,(SE)$

Carpet Plus — $E^6\,N^4$.

Since all three paths lead to the same place, we can write:

$$E^2\,(NE)^4 = (NE)^5\,(SE) = E^6\,N^4.$$

In a sense, the last expression is the simplest.

15. What are the rules that allow you to simplify expressions using the N, E, W, S notation? Explain.