LESSON 8.6

Comparing Populations

EXPONENTIAL GROWTH

Three populations of bacteria are being grown in a laboratory. At time 0: Population A had 10 bacteria; Population B had 100 bacteria; and Population C had 300 bacteria. All three double every hour.

1. Complete the table below to show how the three populations increase as a function of time for the first six hours of growth after time 0.

Population

Time	A	B	C
0	10	100	300
1			

The populations are doubling, which means they are being repeatedly multiplied by 2. Powers of 2 provide a good shorthand for writing the populations.

2. Make another table of the populations of A, B, and C for the first six hours of growth after time 0. This time, use multiplication and a power of 2 to write each population. (Example: For A, the population after four hours is $10 \cdot 2^4$.)

3. Write the expressions for the populations of A, B, and C after:

a. x hours;

b. $x + 3$ hours.

Definitions: This kind of growth is called *exponential growth.* Exponential growth involves *repeated multiplication* by a number. To describe exponential growth, we specify the *starting population* and the *rate of growth.*

For example, if the starting population is 4 and the population triples every hour, this table shows how the population changes as a function of time.

Time	Population	Exponential Expression
0	4	$4 \cdot 3^0$
1	$4 \cdot 3 = 12$	$4 \cdot 3^1$
2	$4 \cdot 3 \cdot 3 = 36$	$4 \cdot 3^2$
3	$4 \cdot 3 \cdot 3 \cdot 3 = 108$	$4 \cdot 3^3$
x	$4 \cdot 3 \cdot ... = ?$	$4 \cdot 3^x$

Generalizations

4. Write an expression for the population after six hours of growth

a. if the starting population is 100 and the population is tripling every hour;

b. if the starting population is 100 and the population is being multiplied by r every hour;

c. if the starting population is p and the population is being multiplied by r every hour.

5. Write an expression for the population after x hours of growth for each situation in problem 4.

SAME POPULATION, DIFFERENT TIME

6. The population of B after five hours is $100 \cdot 2^5$.
   a. Find the population of B at 8 hours, 11 hours, and 14 hours. By how much is the population being multiplied over each three-hour period?
   b. Compare the population of B after x hours with its population after $x + 3$ hours by simplifying this ratio.
   $$\frac{10 \cdot 2^{x+3}}{10 \cdot 2^x}$$

7. The population of A at six hours is $10 \cdot 2^6$.
   a. Compare the population of A after x hours with its population after $x + 5$ hours by simplifying this ratio.
   $$\frac{10 \cdot 2^{x+5}}{10 \cdot 2^x}$$
   b. Check your answer to part (a) by comparing the population of A at 6 hours, 11 hours, and 16 hours.

8. Simplify these ratios.
   a. $\frac{400 \cdot 2^7}{400 \cdot 2^3}$ b. $\frac{100 \cdot 2^{15}}{100 \cdot 2^8}$

9. Simplify these ratios. It may help to substitute values for x and look for a pattern.
   a. $\frac{400 \cdot 2^{x+7}}{400 \cdot 2^x}$ b. $\frac{100 \cdot 2^{3x}}{100 \cdot 2^x}$

10. Solve for x. $\frac{35 \cdot 2^{x+6}}{35 \cdot 2^x} = 2^x$

DIFFERENT POPULATIONS, SAME TIME

11. a. Use the tables you made in problems 1 and 2 to compare the size of A with the size of B at several times. In each case, B is how many times as large? Does this ratio increase, decrease, or remain the same as time goes on?
    b. Repeat part (a), comparing C with B.

12. Simplify these ratios.
    a. $\frac{400 \cdot 2^x}{200 \cdot 2^x}$ b. $\frac{10^0 \cdot 2^{x+4}}{500 \cdot 2^{x+4}}$

13. Solve for x. $\frac{300 \cdot 2^a}{x \cdot 2^a} = 30$

DIFFERENT POPULATIONS, DIFFERENT TIMES

14. Compare these populations using ratios.
    a. B at 10 hours and A at 3 hours
    b. C at 3 hours and A at 6 hours
    c. A at 12 hours and B at 7 hours
    d. C at 1/2 hour and A at 1 hour

15. Compare these populations using ratios.
    a. B at x hours and A at $x + 2$ hours
    b. C at h hours and A at $2h$ hours
    c. A at h hours and B at $h - 5$ hours

16. Simplify these ratios.
    a. $\frac{400 \cdot 2^{x+4}}{25 \cdot 2^x}$ b. $\frac{10 \cdot 2^{4x}}{150 \cdot 2^x}$

17. Solve for x.
    a. $\frac{30 \cdot 2^{a+4}}{x \cdot 2^a} = 60$
    b. $\frac{300 \cdot 2^{a+3}}{x \cdot 2^a} = 24$

POPULATION PROJECTIONS

In 1975 the population of the world was about 4.01 billion and was growing at a rate of about 2% per year. People used these facts to project what the population would be in the future.

18. Copy and complete the table, giving projections of the world's population from 1976 to 1980, assuming that the growth rate remained at 2% per year.

Year	Calculation	Projection (billions)
1976	4.01 + (0.02)4.01	4.09

19. Find the ratio of the projected population from year to year. Does the ratio increase, decrease, or stay the same?

20. There is a number that can be used to multiply one year's projection to calculate the next. What is that number?

21. Use repeated multiplication to project the world's population in 1990 from the 1975 number, assuming the same growth rate.

22. Compare your answer to problem 21 with the actual estimate of the population made in 1990, which was about 5.33 billion.

a. Did your projection over-estimate or under-estimate the 1990 population?

b. Was the population growth rate between 1975 and 1990 more or less than 2%? Explain.

23. At a growth rate of 2% a year, how long does it take for the world's population to double?

REVIEW *FACTORING COMPLETELY*

Example: $16 - 4x^2$ is a difference of two squares, so it can be factored:

$$(4 - 2x)(4 + 2x).$$

However, each of the binomials can be factored further, like this:

$$2(2 - x) \cdot 2(2 + x) = 4(2 - x)(2 + x)$$

Here is another way to factor the same expression:

$$4(4 - x^2) = 4(2 - x)(2 + x).$$

The final expression is the same one we got using the first method. It cannot be factored any further, so we say we have *factored completely*.

Factor each expression completely.

24. $3t^2 - 27s^2$

25. $5x^2 - 180$

26. $x^3y - xy^3$