LESSON 14.5

A Famous Formula

You will need:

- graph paper
- graphing calculators (optional)

STANDARD FORM OF A QUADRATIC

Definition: A quadratic equation is said to be in *standard form* if it is in the form:

$$ax^2 + bx + c = 0.$$

In Chapter 13 you learned several methods to solve quadratics in the case where $a = 1$. If you divide all the terms of a quadratic equation in standard form by a, you can solve it with those methods.

Example: Solve $3x^2 + 5x - 4 = 0$.
Divide both sides by 3:

$$x^2 + \frac{5}{3}x - \frac{4}{3} = \frac{0}{3}$$

$$x^2 + \frac{5}{3}x - \frac{4}{3} = 0.$$

Since $a = 1$, the solutions are $H \pm \sqrt{-V}$. In this case:

$$H = -b/2 = -5/6.$$

Find V by substituting H for x in the equation.

$$V = \left(\frac{-5}{6}\right)^2 + \left(\frac{5}{3}\right)\left(\frac{-5}{6}\right) - \frac{4}{3}$$

$$= \frac{25}{36} - \frac{25}{18} - \frac{4}{3}$$

$$= \frac{25}{36} - \frac{50}{36} - \frac{48}{36}$$

$$= \frac{-73}{36}$$

So the solutions are:

$$-\frac{5}{6} + \sqrt{\frac{73}{36}} \text{ or } -\frac{5}{6} - \sqrt{\frac{73}{36}}$$

The two solutions can be written as one expression:

$$-\frac{5}{6} \pm \sqrt{\frac{73}{36}}$$

where the symbol $\pm$ is read *plus or minus*. It is also possible to write it as a single fraction:

$$-\frac{5}{6} \pm \sqrt{\frac{73}{36}} = -\frac{5}{6} \pm \frac{\sqrt{73}}{6} = \frac{-5 \pm \sqrt{73}}{6}$$

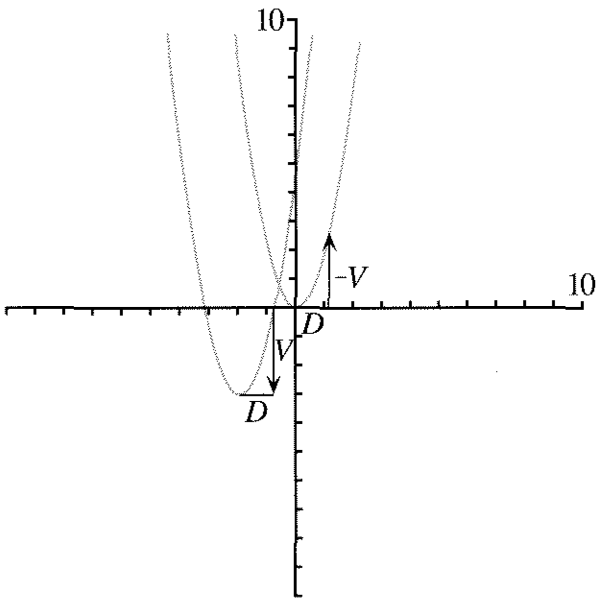
Solve. (Hint: You may divide by a, then use any of the methods from Chapter 13.)

1. $2x^2 + 4x - 8 = 0$
2. $-x^2 + 4x + 8 = 0$
3. $3x^2 + 4x - 4 = 0$
4. $-3x^2 + 8x + 8 = 0$

FINDING THE x-INTERCEPTS

You already know how to find the vertex of a quadratic function in standard form. In this section you will learn how to find the x-intercepts from the vertex.

The following figure shows the graph of the function $y = ax^2 + bx + c$, which is a translation of $y = ax^2$, whose graph is also shown. The coordinates of the vertex are (H, V). D is the distance from the x-intercepts to the axis of symmetry. When $a = 1$, we found that $D = \sqrt{-V}$. What is D in the general case?



The figure shows D and V on a parabola that was translated from $y = ax^2$. In this example, V was a negative number, and the translation was in a downward direction. The arrows representing D and V are also shown on the original parabola. (On $y = x^2$, the direction of the arrow for V was reversed. What is shown is actually the opposite of V. This is indicated by the label $-V$. Since V is negative, $-V$ is positive.)

5. Use the figure to explain why $-V = aD^2$.

6. Express D in terms of V and a.

7. This formula is different from the one we had found in the case where $a = 1$. Explain why this formula works whether $a = 1$ or $a \neq 1$.

SOLVING QUADRATIC EQUATIONS

The x-intercepts, when they exist, are equal to $H \pm D$. It follows from the value of D found in the previous section that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by the formula:

$$H \pm \sqrt{-\frac{V}{a}}.$$

Therefore, one way to solve a quadratic equation in standard form is first to find H and V. In Lesson 2 you learned how to express H in terms of a and b. Then V can be found by substituting H into the equation.

Example: Solve $2x^2 + 8x - 7 = 0$.

Solutions:

$$H \pm \sqrt{-\frac{V}{a}} = -2 \pm \sqrt{-\frac{-15}{2}} = -2 \pm \sqrt{7.5}$$

Solve.

8. $2x^2 + 6x - 8 = 0$

9. $-x^2 + 6x + 8 = 0$

10. $3x^2 + 6x + 1 = 0$

11. $-3x^2 + 6x + 8 = 0$

THE QUADRATIC FORMULA

As you know, $H = -b/(2a)$. The following problem uses that fact to find a formula for V in terms of a, b, and c.

12. Substitute $-b/(2a)$ into $ax^2 + bx + c$ to find the y-coordinate of the vertex as a single fraction in terms of a, b, and c.

If you did problem 12 correctly, you should have found that:

$$V = \frac{-b^2 + 4ac}{4a}.$$

13. To find a formula for the solutions of the quadratic equation in standard form in terms of a, b, and c, substitute the expressions for H and V into the expression

$$H \pm \sqrt{-\frac{V}{a}}.$$

If you did this correctly, you should have found that the solutions are:

$$-\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

14. Show that this simplifies to:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This expression is the famous *quadratic formula.* It gives the solutions to a quadratic equation in standard form in terms of a, b, and c. You will find it useful to memorize it as follows: "The opposite of b, plus or minus the square root of b squared minus $4ac$, all over $2a$."

Solve these equations. (If you use the quadratic formula, you are less likely to make mistakes if you calculate the quantity $b^2 - 4ac$ first.)

15. $2x^2 + 6x - 4 = 0$
16. $-x^2 + 6x + 4 = 0$
17. $3x^2 + 6x - 4 = 0$
18. $-3x^2 + 7x - 4 = 0$
19. Report What are all the methods you know for solving quadratic equations? Use examples.

DISCOVERY *A TOUGH INEQUALITY*

On Friday night when Mary and Martin walked into the G. Ale Bar, Ginger gave them a challenging inequality. "This stumps some calculus students," she said, "but I think you can figure it out."

20. Solve Ginger's inequality: $3 < 1/x$. Check and explain your solution.

REVIEW *RECTANGLES*

21. The length of a rectangle is 10 more than the width. Write a formula for:
    a. the width in terms of the length;
    b. the area in terms of the length;
    c. the perimeter in terms of the width.
22. A rectangle has width $3x + 1$ and length $6x + 2$. Find the perimeter when the area is 200.