LESSON 14.6

Translations of $y = ax^2$

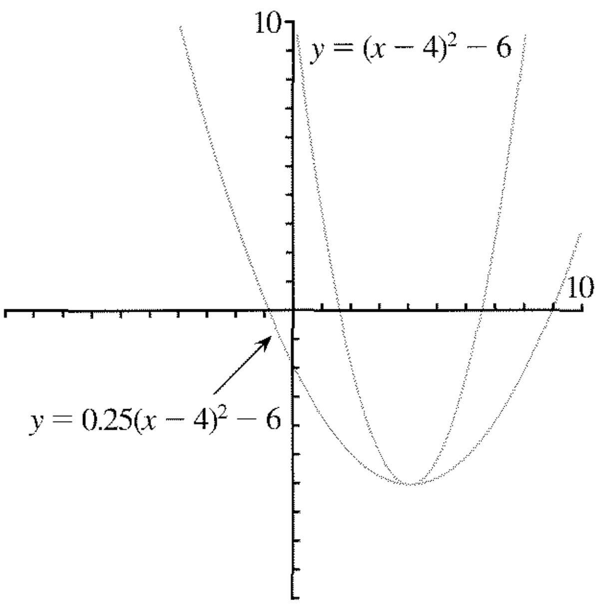
You will need:

graph paper

graphing calculators (optional)

VERTEX FORM

In Chapter 13 you learned that the parameters H and V in the equation $y = (x - H)^2 + V$ represent the coordinates of the vertex of a parabola which is a translation of the one with equation $y = x^2$. This is easy to generalize to any equation in the form $y = a(x - H)^2 + V$, even when $a \neq 1$.



The two parabolas shown in the figure have the same vertex.

1. Write the equation of a parabola having the same vertex as both in the figure that is
   a. more open than either;
   b. more closed than either;
   c. between the two.

2.
   a. Explain why the lowest value for the quantity $(x - 4)^2$ is 0.
   b. Explain how it follows that the lowest point for both parabolas must be for $x = 4$.

3. Write the equation of the parabola that has the same shape as $y = 0.25x^2$ having vertex (-3, 2).

4. Find the equation of a parabola that is a translation of $y = 5x^2$ having vertex (4, -2).

5. The following questions are about the function $y = 6(x + 5)^2 - 4$.
   a. What are the coordinates of the vertex of its graph?
   b. What is the equation of the parabola of the same shape having the vertex at the origin?
   c. What is the equation of the frown parabola having the same shape, and the vertex at the origin?
   d. What is the equation of the frown parabola having the same shape and vertex?

6. **Summary** What do you know about the shape and vertex of the graph of $y = a(x - H)^2 + V$?

MORE ON EQUAL SQUARES

Use the equal squares method to solve each equation. Notice how the solutions of the first equation in each pair differ from the solutions of the second equation.

7. a. $x^2 - 9 = 0$
   b. $4x^2 - 9 = 0$

8. a. $x^2 - 6 = 0$
b. $9x^2 - 6 = 0$

9. a. $(x - 3)^2 - 5 = 0$
b. $16(x - 3)^2 - 5 = 0$

10. a. $(x + 2)^2 - 7 = 0$
b. $3(x + 2)^2 - 7 = 0$

11. **Generalization**

a. Describe how the roots of the second equation in each pair differ from the roots of the first equation.

b. Use the equal squares method to find a general formula for the solutions of the equation $a(x - H)^2 + V = 0$. Explain.

If you did problem 11 correctly, you should have found the same formula as in Lesson 5.

$$H \pm \sqrt{-\frac{V}{a}}$$

COMPLETING THE SQUARE

You can change a quadratic equation from standard form to vertex form by completing the square. When $a \neq 1$, it is more difficult, but it can still be done.

Example: Write $y = 3x^2 + 6x - 9$ in vertex form.

Then complete the square for the quantity inside the parentheses:

$$y = 3(x^2 + 2x + 1 - 1 - 3)$$

Finally, distribute the 3:

$$y = 3(x + 1)^2 - 12$$

So $H = -1$ and $V = -12$. You can check that this was done correctly by finding H and V using the method from Lesson 4:

$$V = 3(-1)^2 + 6(-1) - 9 = -12$$

The same method for completing the square is used even when a is not a common factor.

Example: Write $y = 3x^2 + 5x - 7$

Factor the 3:

$$y = 3\left(x^2 + \frac{5}{3}x - \frac{7}{3}\right)$$

Complete the square:

$$y = 3\left(x^2 + \frac{5}{3}x + \frac{25}{36} - \frac{25}{36} - \frac{7}{3}\right)$$

$$= 3\left(\left(x + \frac{5}{6}\right)^2 - \frac{109}{36}\right)$$

Distribute the 3:

So $H = -5/6$ and $V = -109/12$.

12. Check that H and V were found correctly.

Complete the square.

13. $y = 3x^2 + 6x + 9$

14. $y = -2x^2 + 5x + 8$

15. $y = 2x^2 - 5x + 3$

THE QUADRATIC FORMULA, AGAIN

Let us write $y = ax^2 + bx + c$ in vertex form by completing the square.

Factor the a:

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Complete the square:

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a^2}\right)$$

Distribute the a:

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a}$$

So $H = \frac{-b}{2a}$, and $V = \frac{-b^2 + 4ac}{4a}$ as we saw in Lesson 5.

Finally, if we solve the equation

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a} = 0$$

by the equal squares method, we get:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

So:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

DISCOVERY *EGYPTIAN FRACTIONS*

The ancient Egyptians used only those fractions having 1 for the numerator.

16. Find the sum. Look for patterns.

a. $\frac{1}{5} + \frac{1}{20} = \frac{1}{?}$ b. $\frac{1}{3} + \frac{1}{6} = \frac{1}{?}$

c. $\frac{1}{4} + \frac{1}{12} = \frac{1}{?}$

17. Use the above pattern to predict these missing denominators.

a. $\frac{1}{7} + \frac{1}{?} = \frac{1}{6}$ b. $\frac{1}{?} + \frac{1}{30} = \frac{1}{5}$

c. $\frac{1}{10} + \frac{1}{90} = \frac{1}{?}$

18. Write three more problems having the same pattern as above.

19. Generalization

a. Write an algebraic statement to describe the pattern you found in #16. Use expressions in terms of D for m and n in the equality.

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{D}$$

b. Use algebra to check that your statement is an identity.

20. Find x. Look for patterns.

a. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{x} + \frac{1}{x}$

b. $\frac{1}{4} + \frac{1}{5} + \frac{1}{20} = \frac{1}{x} + \frac{1}{x}$

21. Use the above pattern to express the following fractions as a sum of Egyptian fractions. Check your answers.

a. $\frac{2}{5}$ b. $\frac{2}{7}$

22. Generalization

a. Write an algebraic statement to describe the pattern.

b. Use algebra to check that your statement is an identity.