Arithmetic Sequences and Series

These lessons constitute an informal introduction to arithmetic sequences and series. If you use it with middle school students or in Algebra 1, consider it a preview of the subject. In Algebra 2, think of it as an introduction. The role of formulas depends on the level: with younger students, avoid them; with older students, introduce them with caution.

There is also a little work with mean and median.

The lessons and these notes are slightly edited from *Algebra: Themes, Tools, Concepts*, by Anita Wah and Henri Picciotto. More information on this book can be found at:

http://www.picciotto.org/math-ed/attc.html

1. Staircase Sums

Core Sequence: 1-3, 7-9

Suitable for homework: 4-6, 9

Useful for Assessment: 6

What this Lesson is About:

- a geometric approach to arithmetic sequences
- review of addition of signed numbers
- preview of sums of arithmetic sequences

This is the first of several lessons about sequences. In Algebra 1, we teach them primarily to give students a better understanding of concepts that are more central to the course, such as variables, operations, and functions. They also provide an interesting environment for problem solving, and a preview of future courses.

This lesson previews arithmetic sequences using a geometric approach: students build "staircases" to explore sums of consecutive numbers. Graph paper may be sufficient for most students, but if some want to build the staircases with Lab Gear blocks, allow them to.

One Step at a Time

#2b may be very difficult for some students. You may want to pose the question and come back to it after students have done #3.

#3 is often assigned as a “problem of the week”. It is a rich problem, which can be solved at many levels. Most students should be able to see that odd numbers can be made into two-step staircases, and that powers of two cannot be made into staircases. You can leave it at that, and skip #4-6.

But there is more to find out, and if you wish, you can take a whole class period or more to lead the class’s research into this problem.

For one thing, the problem can be generalized by including negative numbers and 0, as is suggested in #4-6. The number of different ways to write a number as a sum of consecutive integers is equal to the number of its odd factors (which throws light on the difficulty with powers of two.) Students are not likely to discover this on their own, but you can point them in that direction.

Finally, one can think about staircases as the sum of a triangular number, and a multiple of the number of steps. For example, the staircase pictured at the beginning of the lesson is the sum of the fourth triangular number (10 =1+2+3+4), and the number 4 (=4·1). This geometric insight allows us to test whether a number can be written as a sum of n consecutive positive integers by subtracting the n^{th} triangular number from the original number, and seeing if the result is a multiple of n. This is a very sophisticated insight, which you cannot expect to originate among the students.

You could ask students to write a full report on #3, or perhaps to present group oral reports. Don't expect students to come up with all the patterns described above, but do encourage students to look for more patterns than those that are most immediately obvious.

Sums from Rectangles

This is a geometric approach to sums of arithmetic sequences with common difference 1. The very same method works for other cases if all numbers are positive whole numbers.

2. Gauss's Method

Core Sequence: 1-3

Suitable for homework: 4-6

Useful for Assessment: 2, 3

What this Lesson is About:
- an algorithmic approach to arithmetic sequences
- preview of sums of arithmetic sequences

It is important that you not rush to a formula. The methods presented here and in the previous section constitute the mathematical foundation of the formula, and using them deepens the students' understanding of the problem. If your students do not have the mathematical maturity to really understand how the formula is a generalization of the process presented here, teaching the formula is premature. The unfortunate result of teaching a formula too early is that students turn off their brains.

Page Numbers

These problems can be used to preview arithmetic sequences. The problems are not easy, and would make good "problems of the week," that students could work on in conjunction with the next lessons.

3. Sequences

Core Sequence: 1-12

Suitable for homework: 6-12

What this Lesson is About:
- looking for patterns
- using symbolic notation
- definition of a sequence
- even and odd numbers

Graphs of Sequences

More work on graphing sequences can be found in the Geometric Sequences and Series packet.

#3 shows the sequence of triangular numbers, which students may remember from previous work. If students have trouble with #3b, you may hint that this sequence can be seen as a sequence of staircase sums, so the methods from lessons 1 and 2 can be used.

Getting Even
That's Odd

The main point of these sections is once again to give students a chance to recognize numerical patterns and to generalize them with the help of algebraic notation.

#5d may be difficult for some students. As a hint, you may suggest they compare this table to the one for #3. The other patterns are straightforward.

4. Arithmetic Sequences

Core Sequence: 1-6

Useful for Assessment: 6

What this Lesson is About:
- definition of arithmetic sequence

You may explain that arithmetic sequences are a generalization of the staircases studied in a previous lesson. Staircases are arithmetic sequences with common difference 1.

#2-5 should be done in class. #3c in particular would make for a good class discussion. Collect on the chalkboard or overhead student-created sequences for which they did find a formula for the n^{th} term. Check that the formulas do work, and ask the students who discovered them to share their method with the class. If that is insufficient help for everyone to see the pattern, you may point out that in all cases, the formula is of the form $t_n=a+n\cdot d$, where d is the common difference, and a depends on the problem. Finding a is the challenge.

In the course of the discussion, you may discuss the merits of calling the initial term of the sequence t_0 instead of t_1. (Calling it t_0 has the advantage that it is easier to find a formula for t_n.)

Another Odd Triangle

This is an interesting pattern, which touches on many of the ideas in this lesson, and previews the next lesson. It is another candidate for problem of the week.

5. Averages and Sums

Core Sequence: 1-12

Suitable for homework: 7-12

Useful for Assessment: 6, 8, 9, 12

What this Lesson is About:
- Mean and median
- Sums of arithmetic sequences

Means and Medians

In an arithmetic sequence, the mean equals the median. This is also true of any sequence of numbers where the terms are distributed symmetrically around the median.

The main emphasis of the lesson is not on the statistical analysis of data, but on gaining more familiarity with arithmetic sequences.

Means and Sums

The relationship:

"number of terms" times "mean" equals "sum"

is true of any set of numbers. In the particular case of arithmetic sequences, the mean is easy to find: it is the average of the least and greatest terms. This leads to a shortcut in figuring out the sum of an arithmetic sequence.

Again, we avoid emphasizing a formula, and instead encourage students to use reasoning. In some cases, they will need to be able to find the value of the last term, using the technique learned in the previous lesson.

6. More Practice

Theater Seats

This is a "real world" application of sums of arithmetic sequences. It should not present any difficulties.

Equations

Even though the word is not mentioned, these problems give students another opportunity to think about the mean.

Variable Staircases

This section does lead students to discover a formula. But do not encourage them to memorize it. Formulas are difficult to remember, while the methods presented in this packet, once understood, are difficult to forget. In any case, the formula students discover does not have enough generality to deserve memorization, since it only deals with arithmetic sequences with common difference 1.

7. Pyramids

These are not actual 3D pyramids, but rather triangular arrangements involving color.