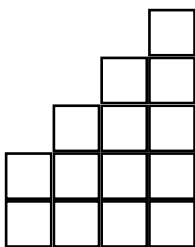# 1. Staircase Sums

You will need: Lab Gear and/or graph paper.

One Step at a Time

Here is an example of a kind of arrangement that we'll call a "staircase". It has 4 steps and the first step is of height 2.

Definition: For this lesson, we will define a staircase as a sequence of stacks of tiles in which each stack is one tile higher than the previous stack. *There must be two or more steps in the staircase*, and the first step can be of any height.

1. How many tiles would you need to build each of these staircases:
    a. First step: 7. Number of steps: 8
    b. First step: 8. Number of steps: 7
    c. First step: 6. Number of steps: 9

2. There are two different 9-tile staircases: 2 + 3 + 4 and 4 + 5.
    a. Find three different 15-tile staircases.
    b. *Find four different 105-tile staircases.

3. **Exploration.** Find every possible staircase with each number of tiles from 2 to 34. Hints:
    •   Work with other students.
    •   Keep organized records of your work.
    •   It is not necessary to draw the staircases.
    •   Look for strategies: what numbers can be made into two-step staircases? three-step?
    •   Look for patterns: what numbers are easiest? what numbers are impossible?

4. The number 10 can be written as the sum of four consecutive numbers.
    a. What are these four numbers?
    b. If negative integers and zero are allowed, can the number 10 be written as the sum of consecutive numbers in any other way? If so, show how.

5. Show how the number 4 can be written as a sum of consecutive integers if negative numbers and zero can be used.

6. **Generalization.** What is the maximum number of consecutive integers that can be used to write the number 17 as a sum? What is the maximum number of consecutive integers that can be used to write the number N as a sum? (Assume N is an integer.) Explain your answer, giving examples.

Sums from Rectangles

7. a. On graph paper, sketch the staircase illustrated at the beginning of the lesson. Then, make a rectangle by sketching a copy of the staircase *upside down* on the first staircase. (You can also do this by building the staircases with tiles or blocks.)
    b. What are the length, width, and area of the rectangle?

8. Imagine a staircase with 100 steps, and a first step of height 17.
    a. It would be half of what rectangle? (Give the length and width.)
    b. How many tiles would you need to build the staircase? Explain how you know.

9. Show how you could find the sum of
    a. the integers from 5 to 55, inclusive
    b. the integers from 0 to 100, inclusive

2. Gauss's Method

Math teachers like to tell a story about Carl Friedrich Gauss. One day in elementary school he was "punished" by his teacher who asked him to add up all the whole numbers from 0 to 100. Carl immediately gave the answer, to his teacher's amazement. He grew up to be one of the greatest mathematician of all time.

Gauss's method was to imagine all the numbers from 0 to written from left to right, and directly beneath that, all the numbers written from right to left. This would look like this:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9\ldots$$
$$100 \quad 99 \quad 98 \quad 97 \quad 96 \quad 95 \quad 94 \quad 93 \quad 92 \quad 91\ldots$$

He mentally added each column, getting 100 each time. He multiplied 100 by the number of columns, and did one more thing to get the correct answer.

1. Finish Gauss's calculation. Be sure to use the correct number of columns, and to carry out the final step. Did you get the same answer as in problem 9b?

2. ∈ What would happen if the numbers to be added started at 1 instead of 0? Obviously, the sum should be the same. Would Gauss's method still give the same answer? Explain.

3. **Summary.** You now know two methods for calculating staircase sums: one involves making a rectangle, the other is Gauss's method. Both methods work well, but it is easy to make mistakes when using them. Write a paragraph explaining how you would use each method to calculate the sum: 5 + 6 + 7 + … + 89. Use sketches as part of your explanation. Both methods should give the same answer.

Page Numbers

4. A book has 518 pages. How many digits are used in numbering the pages of this book? Explain how you figured it out.

5. It took 1992 digits to number the pages of a book. Every page was numbered, starting with page 1. How many pages does the book have?

6. * Explain how to find out how many digits are needed to number the pages of a book that has n pages, if n is:
   a. more than 9, but less than 100
   b. more than 99, but less than 1000

3. Sequences

Definitions: A *sequence* is an ordered list of numbers, called *terms*. (Notice that this is a new use of the word "term".) Terms are often indicated with subscripted variables, such as t_1, t_2, or t_n.

Graphs of Sequences

Definition: The *natural numbers* are the numbers we count with: 1,2,3,4.....etc.
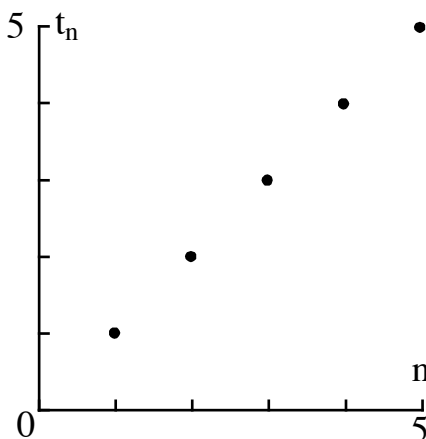
The natural numbers are the easiest sequence of numbers to write using a variable. The first natural number is 1, the second natural number is 2, and so on. t_1=1, t_2=2, … The nth counting natural number is n, so t_n=n. The graph above shows the sequence of natural numbers.

n	1	2	3	4	...	n
t_n	2	5	8	11	...	3n-1

1. Graph the first few terms of the above sequence. Does it make sense to connect the dots? Explain.

2. Make a table and graph the first few terms of the sequence whose n^{th} term is t_n=3n+1. Compare your graph with the one you drew in the previous problem. How are they the same? How are they different?

n	1	2	3	4	...	n
t_n	1	3	6	10	...	?

3. You may remember the above sequence.
   a. What is the 6th term?
   b. *What is the nth term?
   c. Graph the first few terms. Is your graph a straight line?

Getting Even

4. If 2 is the first even number, 4 the second, and so on, what is the millionth even number? In terms of n, what is the nth even number?

5. Graph the first few terms of the sequence of even numbers. Is your graph a straight line?

n	1	2	3	...	?	n
t_n	2	6	?	...	42	?

6. The nth term in the above sequence is the *sum of the first n even numbers*.
   a. What is t_5?
   b. Which term has a value of 42?
   c. Graph the first few terms. Is your graph a straight line?
   d. * In terms of n, what is the nth term of this sequence?

That's Odd!

7. If 1 is the first odd number, 3 the second, 5 the third, what is the one hundredth odd number?

8. a. In terms of n, what is the nth odd number?
   b. Graph the first few terms in the sequence of odd numbers.
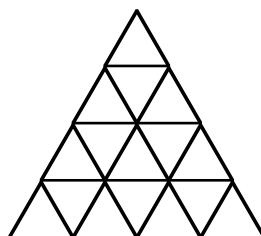
9. a. Look at the figure. How many unit triangles in the first row? the second? the third? (Count triangles whether they point up or down.)
   b. If the triangle were extended indefinitely, how many unit triangles would there be in the nth row?

10. a. How many unit triangles altogether in the first two rows? the first three rows?
    b. How many unit triangles in the first n rows?

11. What is the sum of the first two odd numbers? the first three?

12. a. What is the sum of the first n odd numbers?
    b. Graph the first few terms in the sequence of sums of odd numbers.

4. Arithmetic Sequences

Definition: In an *arithmetic sequence*, the difference between consecutive terms is always the same. It is called the *common difference*.

Examples. These are arithmetic sequences:
	2,7,12,17,22 (the common difference is 5)

5,8,11,14,17,20,23,26,29,32
These are not arithmetic sequences:
	3,9,27,81
	1,-1,1,-1,1,-1
	4,9,16,25,49

1. Which of these are arithmetic sequences? For those that are, what is the common difference?
	a. 2,6,8,12,16,20
	b. 3,6,3,7,3,8
	c. 19, 13, 7, 1, -5, …
	d. the sequence of even numbers
	e. the sequence of odd numbers
	f. 2, 2+9, 2+2·9, 2+3·9, 2+4·9

2. Make up an arithmetic sequence for another student.

3. Answer these questions about a classmate's sequence.
	a. Is it really an arithmetic sequence?
	b. What is the common difference?
	c. * In terms of n, what is the nth term?

4. * For each arithmetic sequence, find the common difference, and write the nth term in terms of n:
	a. 2, 7, 12, 22,…
	b. 2+1·5, 2+2·5, 2+3·5, …
	c. 2, 2+1·5, 2+2·5, 2+3·5, …

5. * Answer the same questions for:
	a. y, y+1·5, y+2·5, y+3·5, …
	b. 2+1·x, 2+2·x, 2+3·x,…
	c. y+1·x, y+2·x, y+3·x, …
	d. y, y+1·x, y+2·x, y+3·x, …

6. **Summary.** Explain how to calculate the nth term of an arithmetic sequence, if you know the first term and the common difference. Test your method on several arithmetic sequences.

Another Odd Triangle

7. Look at the array of numbers.
   a. Write the next two rows.
   b. Describe how the array is made.

$$
\begin{array}{c}
1 \\
3\ 5 \\
7\ 9\ 11 \\
13\ 15\ 17\ 19 \\
21\ 23\ 25\ 27\ 29
\end{array}
$$

8. a. Look at the middle number in rows that have a middle number. What is the pattern?
   b. In rows that do not have a middle number, think of the number between the middle two numbers. What is the pattern?
   c. Find the sum of the numbers in each row. What is the pattern?

9.*a. What is the first number in the nth row?
   b. What is the last number in the nth row?
   c. What is the sum of all the numbers in the first n rows?

5. Averages and Sums

Means and Medians

You probably know how to find the average of a sum of numbers. For example, the ages of the people in Tina's family are 10, 48, 20, 22, and 57. You would find the average age by adding the numbers and dividing by 5.

$$\frac{10 + 48 + 20 + 22 + 57}{5} = 31.4$$

We call this average the *mean.*. Another kind of average is the *median*.

Definition: The *median* is the middle of a set of numbers that are in order from least to greatest. To find the median of an even number of numbers, find the two middle numbers and take their mean.

Examples: To find the median age in Tina's family, first write the numbers in ascending or descending order:

$$10 \quad 20 \quad \mathbf{22} \quad 48 \ 57.$$

The median is **22**.

These are the ages of people in Lana's family: 52, 20, 15, and 53. To find the median, first write the numbers in ascending or descending order:

$$53 \quad \mathbf{52} \quad \mathbf{20} \quad 15.$$

Compute the mean of the middle two numbers :

$$\frac{52 + 20}{2} = \mathbf{36} \text{ , so the median is 36.}$$

1. Find the mean of the set of numbers in each example above. Compare it with the median.

2. Make up a sequence of seven numbers in which:
    a. the mean is less than the median
    b. the median is less than the mean
    c. the mean and the median are equal

3. Repeat the previous problem for a sequence of eight numbers.

4. **Exploration**. Find some sequences of numbers in which the mean and the median are equal. Work with your classmates and compare your answers. What can you conclude about these sequences? Write a summary of your conclusions, including examples. (At least one example should be an arithmetic sequence, and at least one should not.)

5. For each example below, make up two sequences that fit the given description.
    a. The highest term is 19 and both mean and median equal 10.
    b. There are six terms. The greatest is 25, the mean is 10, and the median is less than 10.
    c. There are seven terms. The least is -60, the median is 18, and the mean is less than 18.
    d. The mean and the median are both -4. There are nine terms.

6. ∈ If possible, make up an arithmetic sequence that fits each description in the previous problem. If not, explain why not.

Means and Sums

7. Find the mean and the sum of each arithmetic sequence.
    a. -2,-14,-26,-38,-50,-62,-74
    b. -5,-1.8,1.4,4.6,7.8,11,14.2,17.4
    c. 31,29,27,25,23,21
    d. 18,21,23,26,29

8. € Study your answers to the previous problem.
    a.   In which cases was the mean one of the terms in the sequence?
    b.   When the mean was not one of the terms in the sequence, how was it related to the terms in the sequence?
    c.   How are the number of terms, the mean, and the sum related?

Suppose we wanted to find the sum and the mean of this arithmetic sequence:

$$3 \quad 9 \quad 15 \quad 21 \quad 27 \quad 33 \quad 39 \quad 45 \quad 51$$

Using Gauss's method, write the sequence twice, once from left to right and then from right to left.

$$3 \quad 9 \quad 15 \quad 21 \quad 27 \quad 33 \quad 39 \quad 45 \quad 51$$
$$51 \quad 45 \quad 39 \quad 33 \quad 27 \quad 21 \quad 15 \quad 9 \quad 3$$

9. a. Add each column above.
    b. Find the mean and the sum of the sequence.
    c. € How is your answer related to the sum of each column?

10. Using your results from the previous problem, find a shortcut for calculating the sum and the mean of an arithmetic sequence. Try it on the examples in Problem 7, comparing your results with your previous answers.

11. Find the sum and the mean of each arithmetic sequence described.
    a. The sequence has fifteen terms. The first term is 12 and the last term is 110.
    b. The first term is -11 and the last term is -33. Each term is obtained by adding -2 to the previous term.
    c. The first term is -14 and the difference between consecutive terms is 5. There are forty-one terms in the sequence.
    d. The first term is 7 and each term is obtained by adding -1.4 to the previous term. There are eight terms in the sequence.

12. **Generalization.** Find the sum and the mean of each arithmetic sequence.
    a. The first term is b and the final term is 5. There are six terms in the sequence.
    b. The first term is b and the final term is f. There are 10 terms in the sequence.
    c. The first term is b and the final term is f. There are n terms in the sequence.
    d. * The first term is b and each successive term is obtained by adding d. There are n terms in the sequence.

6. More Practice

Graphing Sequences

1. If you were to plot these sequences (with n on one axis and t_n on the other axis), for which one(s) would the points lie in a straight line? Explain how you know.
    a. 3   3.5   4.5   5.5   6.5   b.   -1   -10   -19   -28   -37   -46
    c. 1/2   1/4   1/8   1/16   1/32   d.   4   7   11   16   22   29

Dominoes

2. Find out what a double-six domino set is. How many dominoes does it consist of? What about a double-nine set? Explain.

Theater Seats

Seats in a theater are arranged so that there are 35 seats in the front (first) row, 38 in the next row, 41 in the row behind that, and so on, adding three seats each time.

3.   How many seats are in the
    a. 10th row?   b.   the nth row?

4. How many total seats are needed if the theater has:
    a. 26 rows?   b.   n rows?

5. How would your answers to the above questions be different if there were 34 seats in the first row?

6. Suppose there were 35 seats in the first row, 37 in the next, and so on, adding two seats each time. How would your answers to the above questions be different?

Equations

7. For each equation, find values of x_1, x_2, and x_3, that make it true.
    a. $(x_1 + x_2 + x_3) / 3 = 100$
    b. $(x_1 + x_2 + x_3) / 3 = 50$
    c. $(x_1 + x_2 + x_3) / 3 = 20$
    d. $(x_1 + x_2 + x_3) / 3 = 10$

8. For each equation in the previous problem, find another set of values for x_1, x_2, and x_3 that will work.

9. If possible, find a value of x_3 to satisfy each equation.
    a. $(15 + 20 + x_3) / 3 = 100$
    b. $(15 + 20 + x_3) / 3 = 50$
    c. $(15 + 20 + x_3) / 3 = 20$
    d. $(15 + 20 + x_3) / 3 = 10$

Variable Staircases

You can build staircases with the Lab Gear. This diagram shows (x) + (x+1) + (x+2) + (x+3)
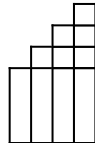
10. In terms of x, what is the sum of (x) + (x+1) + (x+2) + (x+3)?

11. Find the sum of (x)+ (x+1) + (x+2) + (x+3) if:
    a. x = 4   b. x = 99

12. Find each sum. Explain how you got your answer.
    a. (x)+ (x+1) + (x+2) +... + (x+26)
    b. (x+ 1) + (x+5) +...+ (x+84)

13. **Generalization.** What is the sum of each staircase?
    a. 1 + 2 + 3 + … + n
    b. (x) + (x+1) + ... + (x+n)
    c. (x+1) + ... + (x+n)

7. Pyramids

A pyramid is made by stacking rows of blue, red, and yellow blocks. There are 100 blocks in the bottom (first) row 98 in the next row, and so on, with 2 blocks less in each successive row. The bottom row is blue, the next row is red, the third row is yellow and so on, continuing the pattern.

1. Make a sketch or schematic drawing of what you think the pyramid might look like. Write about any patterns you notice.

2. How many rows of blocks are there?

3. How many rows of each color are there?

4. How many blocks are in the 10th row? 11th row? nth row? top row?

5. What color is the 10th row? What color is the top row?

6. There are 30 blocks in a row. Which row is it?

7. Given the number of a row (5th, 10th, 20th, etc.) can you give its color? Explain the pattern.

8. Given the number of a blocks in a row, can you give its color? Explain the pattern.

9. How many blocks total are needed to build the pyramid?

10.   How would your answers to the above questions be different if there were 50 blocks in the bottom row?

11. Suppose four colors were used instead of three. Would any of your answers to the above problems be different? Explain.

12. **Report:** Summarize and explain the patterns you noticed in the above problems. What generalizations can you make?