1. Sequences as Functions

A sequence can be thought of as a function, with the input numbers consisting of the natural numbers, and the output numbers being the terms. In this assignment, we will study sequences as functions.

Definition: In a *geometric sequence*, each terms is obtained from the previous term by mutliplying by a constant amount, the *common ratio*.

Examples: These are geometric sequences.
  2 10 50 250 1250
  3 1 1/3 1/9 1/27

For each of the following,
  a. Tell whether the sequence is geometric, arithmetic, or neither.
  b. If it is arithmetic, find the common difference. If it is geometric, find the common ratio.

1. 5 1 -3 -7 -11

2. -7 2 11 20 29

3. 1 1 2 3 5 8

4. 6 3 3/2 3/4 3/8

5. 25 5 1 1/5 1/25

6. 1/2 3/4 7/8 5/16 31/32 63/64

7. Find the final term of each of these sequences:
  a. a geometric sequence with five terms, common ratio 2, and first term 6
  b. an arithmetic sequence with six terms, common difference 9, and first term -4

8. Find the first term of each of these sequences:
  a. an arithmetic sequence with 10 terms, common difference 7, and last term -3
  b. a geometric sequence with 8 terms, common ratio 1/2, and last term 1/4

9. Graph these arithmetic sequences by graphing the term number (n) on the horizontal axis and the term (t_n) on the vertical axis.
  a. 2 -4 -10 -16 -22
  b. 2 8 14 20 26
  c. -5 -11 -17 -23 -29

10. Graph these geometric sequences.
  a. 2 6 12 24 48
  b. 3 3/2 3/4 3/8 3/16
  c. 1/8 1/4 1/2 1 2

11. These mystery sequences are neither geometric nor arithmetic. Graph them.
  a. 5 8 13 20 29 40 53 68
  b. 7 13 23 37 55
  c. -2 7 22 43 70

12. By looking at the graphs in the previous problem, one might think that the sequences are geometric, but it is clear from looking at the numbers that there is no common ratio. However, the numbers do have a special pattern. Find the pattern and describe it.

13. **Report.** Write a report about what you discovered about graphs of arithmetic sequences, geometric sequences, and the mystery sequences in problem 11. Illustrate your report with examples. Your report should include, but not be limited to, answers to the following questions.
  • Which have graphs that are straight lines? Which have graphs that are curved? How are the two kinds of curved graphs different?
  • For arithmetic sequences, how does the common difference show up in each graph?
  • For geometric sequences, what difference does it make in the graph if the common ratio is greater or less than 1?
  • What are the graphs of the mystery sequences called?

2. Sums of Geometric Sequences

You will need: a yard or meter stick, a ball

The Bouncing Ball

When you drop a ball, it bounces back, but not quite to the height from which you dropped it.

1. Do an experiment where you drop a ball from various heights, and see what height it bounces back to. Use a yard or meter stick to make your measurement. Make a table like this:

Dropped from	Bounced to	Ratio	Difference

2. As you vary the height, what remains closer to constant, the ratio or the difference?

For a certain ideal ball, the bounce height to drop height ratio (or *bounce ratio*) is consistently .8. It is dropped from a height of 2 meters.

3. a. how high does it bounce on the first, second, third bounces?
    b. how many bounces until it bounces to less than 80 centimeters?
    c. how many bounces until it bounces to less than 10 cm?

4. What is the total distance traveled by the ball (both down and up) if someone catches it at the top of its bounce after:
    a. 2 bounces?   b. 20 bounces?

5. Take a guess about the total distance traveled by the ball after 200 bounces. Justify your guess.

Using Symbolic Notation

Say the bounce ratio is r. Then we have:
$$\frac{bounce\ height}{drop\ height} = r$$
Or:
$$bounce\ height = r \cdot drop\ height$$
Assume that the initial drop height is H.

6. How high does the ball bounce on the first, second, third and fourth bounces? Express your answer in terms of H and r.

To analyze the problem of the total distance traveled, it is easier to separate the upward and downward motions. Let us first find the downward distance traveled in the first four bounces:
$$D_4 = H + Hr + Hr^2 + Hr^3$$
As you see, the terms of the sum form a *geometric sequence* with first term H, and *common ratio r*.

7. Write an expression for the downward distance traveled in the first six bounces.

8. € What is the last exponent in the expression for the downward distance traveled in the first n bounces? Explain why the exponent is not the same as the number of bounces.

9. Write an expression for the upward distance traveled in the first four bounces, which you can call U_4.

10. € What is the last exponent in the expression for the upward distance traveled in the first n bounces? Why does this differ from the expression for downward distance?

3. Finding the Sum

Here is a shortcut for calculating the sum of a geometric sequence. We will use the above example of the ideal ball with bounce ratio .8, dropped from a height of one meter, and caught at the top of its fourth bounce. Let us first write the downwards motion:

$$D_4 = 2+2(.8)+2(.8)^2+2(.8)^3$$

Do not calculate the sum! You will soon see why.

Multiplying both sides by .8, we get:

$$D_4(.8)=2(.8)+2(.8)^2+2(.8)^3+2(.8)^4$$

Subtracting one equation from the other:

$$D_4 - D_4 \cdot (.8) = 2 - 2(.8)^4$$

1. ∈ Explain why there are so few terms after subtracting.

2. *Solve* for D_4. (Hint: factor, then divide.)

3. Use the *multiply-subtract-solve* technique to find U_4.

4. What is the total distance traveled by the ball in four bounces?

When adding only four terms, the multiply-subtract-solve technique is not much of a shortcut. However, when adding large numbers of terms, it is extremely convenient. For example, for 20 bounces, you would start by writing:

$$D_{20} = 2+2(.8)+\ldots+2(.8)^{18}+2(.8)^{19}$$

5. ∈ Explain why in this case the last terms do not contribute very much to the sum.

6. Use the multiply-subtract-solve technique to check your answers for problems 4b and 5 in Lesson 2.

7. **Summary**. Summarize what you learned about the sum of geometric sequences.
   a. Explain the multiply-subtract-solve method. (What does one multiply by? What does one subtract? What does one solve for and how?)
   b. What is the effect of the common ratio on the sum? (What if r is less than 1? What if it is equal to 1? What if it is greater than 1?)

8. **Generalization.** Use the multiply-subtract-solve technique for these sums:
   a. $S = a + ar + ar^2 + \ldots + ar^{n-1}$
   b. $S = a + ar + ar^2 + \ldots + ar^n$

4. Decimals and Fractions

Writing Fractions as Decimals

1. How does one convert a fraction to a decimal number? Give examples.

When converting fractions to decimals, sometimes you get a *terminating* decimal, such as 3.4125, and sometimes you get a *repeating* decimal, such as 7.8191919… This last number is often written $7.8\overline{19}$.

The next two problems are easier if you work with lowest terms fractions.

2. **Exploration.** For what fractions do you get a repeating decimal? Does it depend on the numerator or denominator? (Hint: pay attention to the prime factorization of the numerator and denominator.)

3. **Exploration.** For repeating decimals, is there a pattern to how many digits are in the repeating part? What is the longest possible repeating string for a given denominator? (Hints: use long division rather than a calculator to explore this.)

4. * Explain why the decimals obtained as a result of a division *must* repeat or terminate.

5. ∈ Explain why some calculators give a decimal that does not seem to repeat for 2/3: .6666666667

Writing Decimals as Fractions

Example: 3.4125 can be converted to a fraction by multiplying it by 10^4, which gets rid of the decimal, and then dividing by 10^4, which gets us back to the original number:
$$\frac{34,125}{10,000}$$

6. Convert these decimals to fractions:
   a. 6.0   b. 3.2
   c. 0.015   d. 3.41

The case of repeating decimals is more difficult. Take $7.8\overline{19}$. Clearly, it is greater than 7.81, and less than 7.82. So it is between 781/100 and 782/100.

To find a single fraction it is equal to, we can rewrite it as:
$$7.80\overline{19}$$
$$= 7.8 + .0\overline{19}$$
$$= 7.8 + .019 + .00019 + .0000019 + \ldots$$

Observe that:
$$.00019 = .019(.01)$$
$$.0000019 = .019(.01)^2$$

7. Write the next term in the sum as a decimal, and as a product of .019 and a power of .01.

As you see, $7.8\overline{19}$ is the sum of 7.8 and a geometric sequence with first term .019 and common ratio .01. The sum of the first three terms of the geometric sequence can be written:
$$S = .019 + .019(.01) + .019(.01)^2$$
Multiply both sides by .01:
$$S(.01) = .019(.01) + .019(.01)^2 + .019(.01)^3$$
Subtract:
$$S(1 - .01) = .019 - .019(.01)^3$$
Solve:
$$S = \frac{.019 - .019(.01)^3}{.99}$$

Multiplying numerator and denominator by 1000:

$$S = \frac{19 - 19(.01)^3}{990}$$

So:

$$7.8\overline{19} = 7.8 + S$$

$$= 7.8 + \frac{19 - 19(.01)^3}{990}$$

$$= \frac{7.8(990) + 19 - 19(.01)^3}{990}$$

$$= \frac{7741 - 19(.01)^3}{990}$$

$$= \frac{7741 - .000019}{990}$$

The sum is very close to 7741/990.

8. Use the multiply-subtract-solve technique to add:
   a. the first 4 terms
   b. the first 5 terms

9. € The numerator differs from 7741 by $19(.01)^n$ if we add up the first n terms. Explain.

If we use large values for n, we find that the sum can get as close to 7741/990 as we want. (Even with fairly small values, calculators cannot show a difference.) Mathematicians say that the whole infinite sum *converges* to 7741/990, and they agree that we can write an equality:

$$7.8\overline{19} = 7741 / 990 .$$

10. Check that this equality is correct by converting the fraction back to a decimal.

A Shortcut

A quick way to find the fraction is to use the multiply-subtract-solve technique on the decimal itself:

$$S = 7.8\overline{19}$$

$$S = 7.8191919\ldots$$

$$.01S = 0.0781919\ldots$$

Subtract:

$$S - 0.01S = 7.8191919\ldots - 0.0781919\ldots$$

$$(1 - 0.01)S = 7.819 - 0.078$$

(Notice that the infinite sequence of 19s disappeared.)

$$.99S = 7.741$$

$$S = \frac{7.741}{.99} = \frac{7741}{990}$$

11. Convert to a fraction:
    a. $.\overline{65}$ b. $4.3\overline{21}$

5. More Practice

Sums

1. Find each sum.

   a. $\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^{2}+\left(\dfrac{1}{2}\right)^{3}$

   b. $\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^{2}+\left(\dfrac{1}{2}\right)^{3}+\left(\dfrac{1}{2}\right)^{4}$

   c. $\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^{2}+\left(\dfrac{1}{2}\right)^{3}+\left(\dfrac{1}{2}\right)^{4}+\left(\dfrac{1}{2}\right)^{5}$

   d. $\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^{2}+\left(\dfrac{1}{2}\right)^{3}+...+\left(\dfrac{1}{2}\right)^{n}$

2. This sum goes on for ever. (We call it an *infinite series*.) Use the pattern you found in the problem above to estimate the sum of this infinite series:
   $$\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^{2}+\left(\dfrac{1}{2}\right)^{3}+...$$

3. Find the sums of these infinite series.

   a. $\dfrac{1}{3}+\left(\dfrac{1}{3}\right)^{2}+\left(\dfrac{1}{3}\right)^{3}+...$

   b. $\dfrac{1}{4}+\left(\dfrac{1}{4}\right)^{2}+\left(\dfrac{1}{4}\right)^{3}+...$

   c. $\dfrac{1}{k}+\left(\dfrac{1}{k}\right)^{2}+\left(\dfrac{1}{k}\right)^{3}+...$

   (Assume that k is a positive integer.)

Geometric Sequences

4. Some of the following sequences are geometric: find their common ratio. Some are arithmetic: find their common difference.

   a. $2/3, (2/3)^{2}, (2/3)^{3}, ...$

   b. $1/3, 4/3, 7/3, 10/3,...$

   c. $10, 10/8, 10/64, ...$

   d. $10, 80, 640, 5120, ...$

   e. $1/3, 8/3, 64/3, 512/3, ...$

5. Find the sum of the first 50 terms for the sequences above.

6. Two of the sequences above are such that if you add the entire infinite sequence, the sum converges to a finite number.

   a. Explain how you can tell which sequences they are.

   b. Find the sum they each converge to.

Inheritance

The brothers Able and Earl inherited an acre of land from their father, which they divided equally. Each brother willed his land to his family. Able's family was large and Earl's was small. Able's family needed more land, so they bought 40% of the land belonging to Earl's family. In the next generation, Able's family again bought 40% of Earl's family's land. This continued for several generations.

7. Copy and and extend the table below to show the amount of land owned by each family up to the eighth generation.

Generation	Able's land	Earl's land
1	.5	.5
2	.7	.3

8. Study the data. At this rate, will Able's family ever own the whole acre? Explain.

Decimals and Fractions

9. Write as a fraction:

   a. $.\overline{21}$

   b. $.3\overline{21}$

   c. $.\overline{321}$

10. Find whole numbers p and q such that:

   a. $.45 < p/q < .46$

   b. $.\overline{4} < p/q < .45$

Rational Numbers

Definition: A *rational number* is a number that can be written as a fraction with an integer numerator and denominator.

Examples: 7, .5, and -.66666… are rational numbers, because they can be written as 7/1, 1/2, and -2/3.

Show that the following numbers are rational:

11. a. .3
    b. .3333…

12. a. .142857
    b. $.\overline{142857}$

13. a. .0909090…
    b. .9090909…

14. a. .1111111…
    b. .2222222…

15. € Mathematicians believe that .99999… = 1. Explain why.

Bounce Ratios

16. What is the total distance traveled in 200 bounces by a ball with the following bounce ratios, after being dropped from a height of two meters?
    a. A super-ball, with bounce ratio .9.
    b. A flat basketball, with bounce ratio .3.

An absent-minded professor invents a hyper-ball with a bounce ratio of 1.1.

17. Repeat Problem 17 for the hyper-ball.

18. Repeat Problem 17 for a defective hyper-ball with a bounce ratio of only 1.