Geometry of Function Diagrams

Geometry Reminders

Fact P: In a quadrilateral, if a pair of opposite sides is both equal and parallel, then the quadrilateral is a ___________. (You can prove this with the help of congruent triangles.) Opposite sides of a parallelogram are equal.

Fact T: If the corresponding angles determined by a transversal are equal, then the two lines are ___________. If two lines are parallel, then corresponding angles are __________.

Fact AA: If two pairs of corresponding angles in two triangles are equal, then the triangles are __________.

Theorem: In the function diagram for \( y = mx + b \), either all in-out lines are parallel, or they meet in one point

1. On the diagram of \( y = mx + b \) shown on the right, how long are the vertical sides of the quadrilateral?

2. If \( m=1 \), show that the quadrilateral is a parallelogram (and therefore that the in-out lines are parallel.)

Since \( x \) is a generic input, you have proved that in the case where \( m = 1 \), all in-out lines are parallel to the one through \( 0 \), and therefore to each other.

If \( m \neq 1 \), the quadrilateral is not a parallelogram, since opposite sides are unequal, and therefore the two in-out lines meet at a point we will call \( F \).

Strategy for proof: We would like to prove that all the in-out lines go through that same point \( F \), the focus. To do that, we will show that the position of \( F \) on the in-out line \((0, b)\) does not depend on the choice of \( x \).

Consider the diagram below. Let us say that the length \( 0b = z \). This number does not depend on \( x \). (It depends on \( b \), and on how wide apart the axes are.)

4. Show that the triangles \( Fyb \) and \( Fx0 \) are similar, with ratio of similarity \( m \).

It follows that \( \frac{Fb}{F0} = m \), and therefore \( \frac{F0 + 0b}{F0} = m \)

5. Use algebra to show that \( F0 = \frac{z}{m - 1} \) if \( m \neq 1 \)

So \( F \) is in the same position for any input \( x \). In other words, all the in-out lines go through the focus, which is what we wanted to prove.
Theorem: If a function diagram has a focus, then the equation is of the form \( y = mx + b \)

In other words, if all the in-out lines of a function diagram meet in one point \( F \), then the image of all points \( x \) is given by the same formula, in the form \( y=mx+b \).

**Strategy for proof:** we will use similar triangles to help us find a formula for the output corresponding to the input \( x \).

1. Using a ruler, draw in-out lines for 0 and \( x \). Label the output for 0 as \( b \), and the output for \( x \) as \( y \).

Call the ratio \( \frac{Fb}{F0} = m \).

2. Show that the two triangles are similar, with ratio \( m \).

3. Find the ratio of the vertical sides, and solve for \( y \).

Since \( m \) and \( b \) are constants that depend only on \( F \)’s position, and since \( x \) was a completely generic point, we have proved the theorem.

Theorem: If all the in-out lines of a function diagram are parallel, then the equation is of the form \( y = x + b \)

In other words, if all the in-out lines are parallel, then the image of all points \( x \) is given by the same formula, in the form \( y=x+b \).

**Strategy for proof:** we will use a property of parallelograms to help us find a formula for the output corresponding to the input \( x \).

4. Why is the quadrilateral a parallelogram?

5. Use a property of parallelograms to show that \( y = x + b \)

Since \( b \) does not depend on \( x \), and since \( x \) is a completely generic point, we have proved the theorem.