Geometric Puzzles for Prospective Math Teachers

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The Common Core State Standards for Mathematics (CCSSM, [4]) call for incorporating eight mathematical practices throughout K-12 education. The first one is "Make sense of problems and persevere in solving them." This echoes the National Council of Teachers of Mathematics' long-standing emphasis on problem-solving [15], and the Conference Board of the Mathematical Sciences promotion of active learning and inquiry [5]. The third Student Learning Objective (SLO, [13])[1] suggests that these values should be incorporated in geometry courses for prospective teachers (GeT courses.)

By their very nature, geometric puzzles do not lend themselves to a cookbook approach ("Here is the recipe — now practice applying it.") Geometric puzzles are problems to be solved, and provide an excellent vehicle to infuse inquiry into any GeT course. They can be flexibly inserted in the course, and will support several of the SLOs that constitute the basis of this book. GeT students who work on geometric puzzles are practicing the sort of thinking that many of them will be expecting from their own students once they teach geometry in secondary school. The GeT instructor can model how to achieve that.

Well-chosen puzzles for a GeT course share some of these characteristics:
◊ They generate curiosity and engagement.
◊ They provide a break from the everyday classroom routine.
◊ They are solvable by the students, but they are sufficiently challenging to make the solution feel satisfying.

These features describe any good puzzle. Puzzles intended for a GeT course are more constrained.

For one thing, the backgrounds and talents of the students in any one class will vary widely. One student's "easy-peasy!" is another student's "impossible!." One strategy to address this is to present sets of related puzzles at various levels of difficulty. This allows each student to find their own path through the set, and what they learn from the more accessible puzzles will help them develop intuitions and strategies for the more challenging ones. Another advantage of such a multi-puzzle setup is that it facilitates collaboration in a way that a puzzle that depends on a single "aha" does not. It also provides the instructor with a hinting strategy: "Solving this easier puzzle will help you make progress on the one you that is currently frustrating you." (Several of the examples in this chapter consist of such linked puzzles.)

The other constraint for a worthwhile GeT course puzzle is that it must be related to the intended learning outcomes: it must be curricular. A puzzle can provide an introduction to a

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[1] SLO 3 is "Understand the ideas underlying current secondary geometry content standards and use them to inform their own teaching." It will be referenced throughout this chapter.

topic, or it can provide a context for an application of a concept that has already been taught. It is even possible to use a puzzle to be the student's main or only involvement with the underlying mathematics. In fact, there is no need for all puzzles to be inserted into the course in the same way, and it may not even be possible. Different instructors will make their own choices: puzzles can be pursued in class or included in homework; they may involve hands-on materials or online applets; they can be required or optional; and so on.

In my 50 years as a math educator I have used geometric puzzles with students of all ages, Kindergarten to 12^{th} grade, and in professional development workshops for teachers.[2] In this chapter, I share a selection of puzzles which I believe would enhance any GeT course.[3] I start with tessellation, the tiling of the plane, because it supports a discussion of very basic concepts of high school geometry and provides a context for an introduction to geometric transformations (a topic that may be unfamiliar to many GeT students.) The next four puzzles are dissection puzzles, which involve among other things the isosceles right triangle and the concept of similar figures. I present them in order of difficulty. The final section of this chapter is about the geoboard, a hands-on environment where geometric puzzles can be posed to support the Pythagorean theorem, the manipulation of radicals, and other high school geometry topics.

Tessellation

The tessellation puzzle is to use congruent copies of a single figure (the tile) to cover the plane with no overlaps and no gaps between the tiles.[4]

Students can be challenged to **find tilings of the plane, using:**
1. **Grid paper or triangle paper**
2. **Triangles**
3. **Quadrilaterals**
4. **Regular polygons**

This is a richly curricular activity. In many cases, there are multiple ways to do it, and students may explore those options. For challenges #2-4, the exploration can take place using interactive geometry software, or, in fact, just about any application where one can create

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[2] I will not be shy about sharing links to curriculum materials I developed for use in high school classes, with the caveat that some may need to be adapted to use with undergraduates.

[3] One style of puzzles I have not used in the classroom, as I came across it after I retired, is exemplified by Catriona Agg's prodigious creations [1].

[4] These activities are spelled out in Chapter 7 of [17], pp. 97-104.

graphics, as long as they make it possible to create regular polygons for #4. Off screen, one can use the geometry templates which are available from math education publishers.[5]

SLO 2 is "Evaluate geometric arguments and approaches to solving problems." It calls for students to "reflect on their own reasoning, share their reasoning with one another, and critique the reasoning of their peers." It can be helpful to initially launch this sort of discourse in a context where the subject under discussion is not seen by the students as textbook content, and is in fact created by the students themselves.

As they find tilings on grid paper or triangle paper,[6] they can be asked to justify the claim that their discoveries extend infinitely in all directions. One common argument suggested by students is that the tiles can be arranged into stripes, and the stripes can be juxtaposed, as in Figure 1.

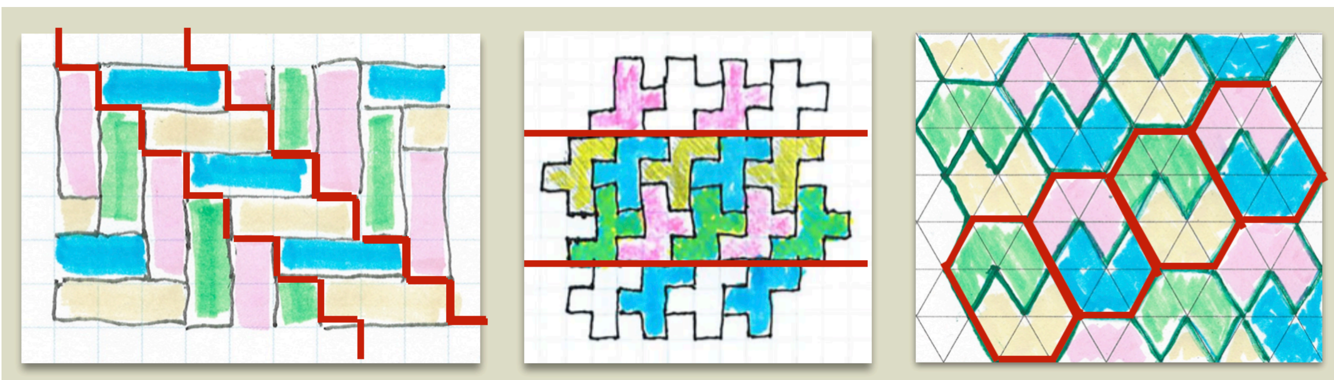

Figure 1.

Geometry Basics

Triangle tilings are easy enough for students to find, whether with "special" triangles (e.g., right or isosceles) or with a generic scalene triangle. Figure 2 shows one such tiling.

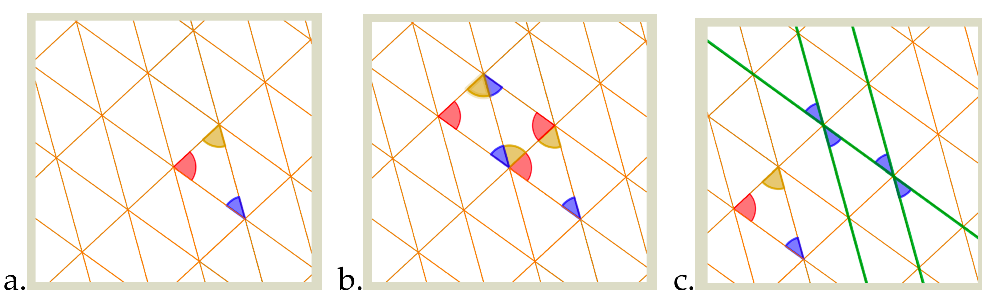

Figure 2.

[5] My *Geometry Labs* Template is specially designed to support these activities, and includes many regular polygons.

[6] mathed.page/space/triangle-paper.pdf

Inspection of this figure can generate discussion about (and suggest proofs of) several basic high school geometry results. I colored some angles in the figure. Note that angles with the same color are congruent, since they are parts of (copies of) the same triangle. By looking at these angles in different locations (figure 2b,) we see that the measures of the angles in a triangle add up to 180°. Other basic facts are revealed: the exterior angle theorem (figure 2b,)[7] the congruence of vertical angles, the equal angles created by parallels and a transversal (figure 2c,) and more.

Having this conversation should bring back high school memories for most GeT students, as these results are part of any national or state geometry standards. SLO 3 calls for deep understanding of such content. Moreover, seeing basic theorems introduced in this way could support another goal of SLO 3: to "foster the construction of pedagogical content knowledge by sharing teaching techniques and by engaging students in conversations about teaching geometry content."

Tiling the plane with parallelograms or other special quadrilaterals is not difficult, but after some unsuccessful attempts, the fact that any quadrilateral at all can tile the plane is surprising to many students. Non-convex quadrilateral tilings, such as the one in figure 3, are particularly intriguing.

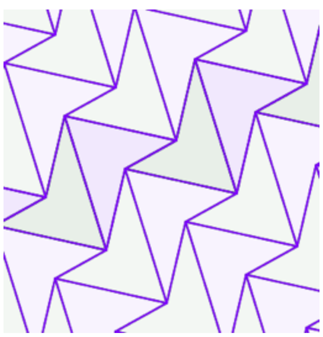

Figure 3.

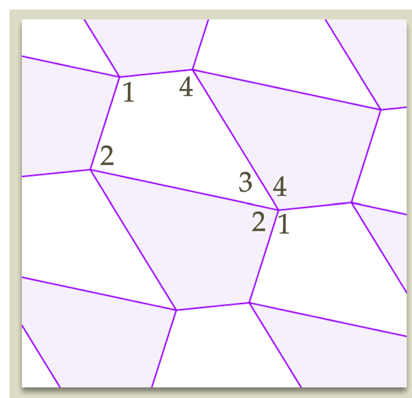

Figure 4.

Figure 4 can help understand why any quadrilateral can tile the plane. It shows that the sum of the quadrilateral's angles is 360°, and thus that all four angles have to meet at each vertex. To accomplish this, the tiles have to be laid so that the corresponding vertices of shared sides are at opposite ends of the line segment they have in common. Following this guideline consistently yields the necessary arrangement.

[7] An exterior angle is the angle formed by a side of a polygon and an extended adjacent side. The exterior angle theorem states that for a triangle, the measure of an exterior angle is equal to the sum of the nonadjacent interior angles.

The fact that only three regular polygons will tile the plane (equilateral triangle, square, regular hexagon) is easy enough to prove, but it naturally leads to our fourth challenge: the search for Archimedean tilings, which involve combining different regular polygons.

Once again, for the tiling to succeed, the sum of angles at a shared vertex needs to be 360°. A proper exploration therefore relies on finding the (interior) angles of regular polygons. Those can be found by dividing the polygons into triangles. Figure 5 shows two ways to do this in the particular case of a regular pentagon.

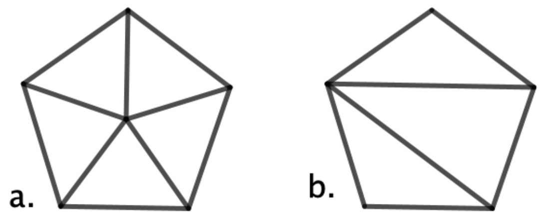

Figure 5.

Both ways are based on the fact that the sum of the angles in any triangle is 180°. In Figure 5a, we have five triangles, so a total of 900°. We should subtract 360° to account for the angles at the center, as they do not contribute to the sum of the interior angles. Each interior angle, therefore, must be (900° – 360°)/5 = 108°. In figure 5b, there are two fewer triangles, but all their angles contribute to the sum of the interior angles. Both approaches lead to the same answer. Either approach can be generalized to a regular n-gon, yielding the same general formula for the measure of the interior angles. That formula can be entered into a spreadsheet to yield the needed angles for various values of n.

Yet another approach to finding the angles is based on exterior angles ("turn angles".) An ant walking around the perimeter of a regular n-gon will have turned a total of 360° after going all the way around. Thus the measure of each turn is $360°/n$, and the corresponding interior angle is $180^\circ - \frac{360^\circ}{n}$. A discussion of these three proofs is an opportunity to get across the idea that there are different ways to get to a theorem, and that each one illuminates a different aspect of the question at hand. This supports SLO 1 ("Derive and explain geometric arguments and proofs.") It states: "Students should understand … that geometric arguments may take many forms. They should understand that in some cases one type of proof may provide a more accessible or understandable argument than another." And also: "It is valuable to show GeT students a range of possibilities and ask them to evaluate the pros and cons as a class."

Trying to tile the plane with regular pentagons and decagons leads to an interesting discovery: at a given vertex, the angles of two pentagons and one decagon do add up to 360°, as desired. Unfortunately, this necessary feature is not sufficient, as attempted tilings capitalizing on this fact always fail. Figure 6 shows one such attempt. This observation is one way to trigger a discussion of the difference between "necessary" and "sufficient", a crucial understanding for future math teachers.

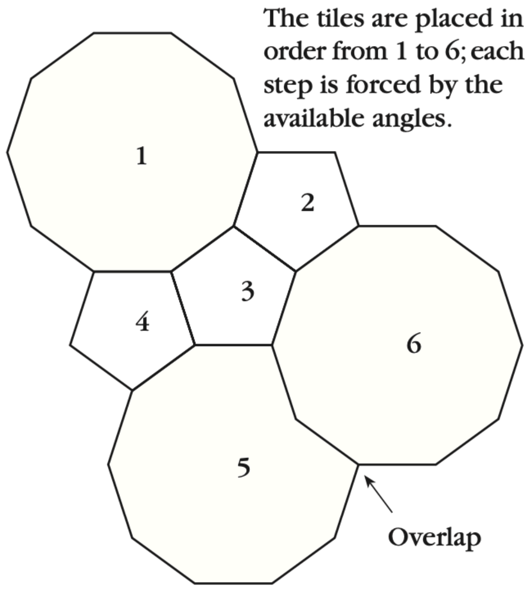

Figure 6.

Transformational Geometry

SLO 10 is "Use transformations to explore definitions and theorems about congruence, similarity, and symmetry." It states: "A GeT course can contain a dedicated unit on transformations, or transformational concepts can be integrated throughout the course." In either case, tilings of the plane are a fertile environment for introducing isometries (known as *rigid motions* in the CCSSM,) and some basic theorems about them.

All periodic tilings include translations in more than one direction, and thus lend themselves to a discussion of the composition of translations. Figure 7 shows basic brick layout.

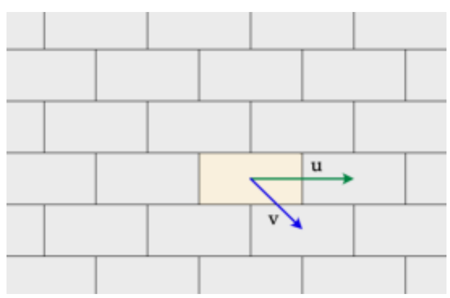

Figure 7.

There are many options for translations that preserve the tiling. In this figure, I chose two that connect the centers of bricks in such a way that you can get from any brick to any other brick by combining translations using vectors $\vec{u}$ and $\vec{v}$ (or their opposites) repeatedly. The composition of translations is a translation by the sum of their respective vectors.

In analyzing periodic tilings, perhaps from Islamic art or Escher prints, students can search for mirror lines and centers of two-fold, three-fold, four-fold, and six-fold rotations. This connection between mathematics and art is quite popular with students. Figure 8 shows two examples, but there is no shortage of such images on the Web. (In fact, students can be asked to find their own examples to analyze.)

Figure 8.

In the case of two-fold rotations, one sees that the composition of two of them (with different centers) is always a translation. In figure 9, M and N are the midpoints of their respective sides. Quadrilateral *1* is rotated around M, which yields quadrilateral *2.* Quadrilateral *2* is rotated around N to create quadrilateral *3,* which turns out to be the image of the original, using the translation $2 \cdot \overrightarrow{MN}$.[8]

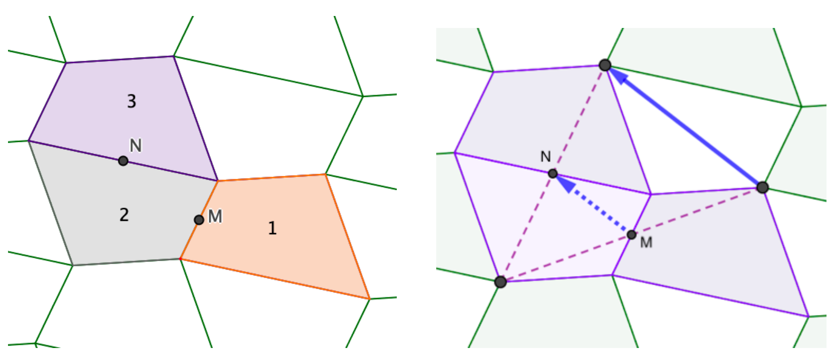

Figure 9.

[8] Basic theorems such as this one, and their proofs based on standard high school geometry theorems can be found in [2] (out of print but findable on the Web.) Formal definitions of the various geometric transformations, and a rigorous all-transformational approach to those theorems is developed in [6]. For worksheets guiding students through many of these ideas, see mathed.page/transformations and *Isometries of the Plane* (mathed.page/transformations/isometries/isometries2.pdf.)

This turns out to be the key to an algorithm for creating quadrilateral tilings. Place the next tile by rotating the current tile 180° around the midpoints of its sides. This will work for any quadrilateral. (In fact the same algorithm can be used to generate triangle tilings.) This supports SLO 10 (transformations — in this case 180° rotation) and SLO 6 (since the algorithm can be applied to generate tilings using interactive geometry software tools.)

A little more surprising to many students is that the composition of two reflections is a translation or a rotation, depending on whether the reflection lines are parallel or not. This is a foundational theorem of transformational geometry, which is unfortunately not included in the CCSSM. Because of its importance, and because teachers need to have a deeper understanding of what they are teaching than their students, it should be included in a GeT course. The most effective way to introduce it is with the help of interactive geometry software (SLO 6,)[9] but tilings such as the ones in figure 10, or the ones found by students, also offer a context for discussion.

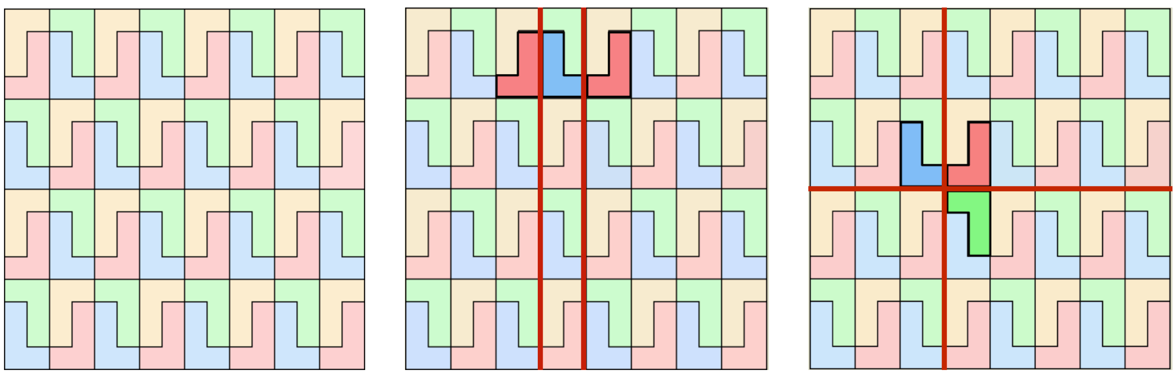

Figure 10.

To summarize: teacher-supplied and/or student-created tilings can be used to preview, launch, or apply material included in SLOs 3 and 10. Some of this content may already appear in existing mathematics courses. A GeT instructor can minimize or avoid duplicating the content of those courses by introducing tessellations as puzzles to be solved, emphasizing their pedagogical use, and prioritizing an informal approach based on student exploration so as to model the Standards for Mathematical Practice of the CCSSM.

[9] SLO 6 is "Effectively use technologies to explore geometry and develop understanding of geometric relationships." It is essential that GeT students work with interactive geometry software as it offers game-changing pedagogical opportunities, but it is not a focus of this chapter.

Tangrams

We now transition from tiling the whole plane to tiling finite polygons. *Tangrams* are perhaps the best-known of all geometric puzzles, and the one with the longest history.[10] The solver is to rearrange the seven pieces shown in Figure 11 to make a square.

Figure 11.

This classic puzzle does not lend itself to curricular use, as it is too time-consuming for some, and familiar to others. However some other puzzles using these pieces do belong in the secondary classroom. They can help with the introduction of basic geometry vocabulary such as the names of various “special” quadrilaterals. Tangram activities can involve measurement, symmetry, and convexity.[11] For the purpose of a GeT course, I recommend these two challenges.

1. **Is it possible to make a square using 1, 2, 3, …, 7 pieces? If yes, record your solution. If not, prove that it is indeed impossible.**

In commercially available tangrams, the length of the small isosceles right triangle’s hypotenuse is two inches. This is not the case, of course, for the online versions of the puzzle,[12] but the mathematics works out the same way. If the puzzles are being solved on a screen, the instructor should just state that the hypotenuse of the small triangle has a length of two units.

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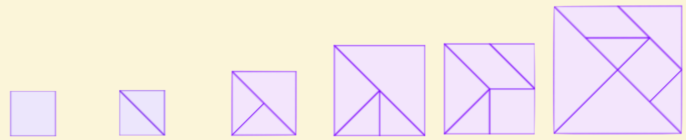
[10] See [10] (chapter 18,) and [12] (chapters 3 and 4.) See also [20].

[11] See chapter 2 (pp. 25-32) and Lab 10.6 (pp. 143-144) of [17].

[12] One such implementation is on my website: mathed.page/geometry-labs/tangrams

From there, students can figure out that the area of each small triangle is 1 square inch, and in fact the measurements of all the pieces follow. Figure 12 shows the possible squares, most of which are sure to be found by students.



Pieces	1	2	3	4	5	7
Area	2	2	4	8	8	16
Side	$\sqrt{2}$	$\sqrt{2}$	2	$2\sqrt{2}$	$2\sqrt{2}$	4

Figure 12.

A six-piece square is not possible. One proof, well within the reach of an undergraduate, is based on the measurements of the tangram pieces. The total area of all the pieces is 16 in^2. Individual pieces have area 1, 2, or 4 in^2, so the area of six pieces must be 15, 14, or 12 in^2. If there were a six-piece square, its side would have to be the square root of one of these numbers, but that cannot be achieved, as the sides of tangram pieces are either whole numbers, or multiples of $\sqrt{2}$.

This supports SLO 1, which states: "…a GeT course should enhance a student's ability to read and write proofs of theorems, apply them, and explain them to others." Also, since the proof is based on measurements of an isosceles right triangle (a *special right triangle*, in school parlance,) the puzzle supports SLO 3, which calls for understanding of "current secondary content standards." In particular, one way to obtain all the measurements is to use Pythagorean theorem. The square root of two will arise in this exploration — another staple of the high school geometry course. In fact, the geometric interpretation of the square root is essential to the impossibility proof.

2. **Is it possible to make a convex tangram *n*-gon for *n* = 3, 4, 5, …? If yes, record your solution. If not, prove that it is indeed impossible. (You do not need to use all seven pieces.)**

Figure 13 shows a solution for $n = 7$, and an incorrect solution for $n = 8$, which was proposed as a joke by a high school teacher.

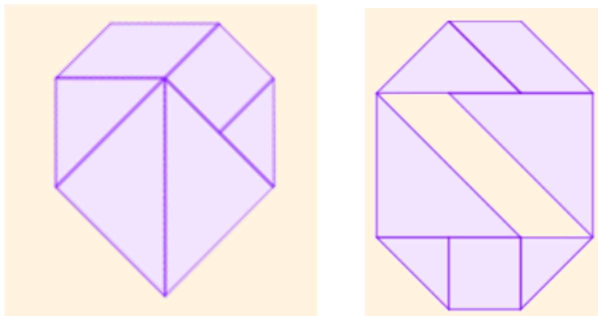


Figure 13.

One proof that it is not possible for $n > 8$ is by way of an exploration of the exterior angles. The interior angles of all tangram pieces are multiples of 45°. The greatest are 135°. Likewise, the greatest possible sum of angles at a vertex is 135°. Therefore the least possible exterior angle is 45°. If an ant were to walk around the perimeter of a convex tangram figure, the "amount of turning" at each vertex would be the exterior angle. Since the total turning after a full circuit would be 360°, and since each turn can be no smaller than 45°, the maximum number of vertices is 8.

The insights conferred by this proof help support one of the parts of SLO 1: "…the role of a proof is not merely to show that something is true but to clearly communicate to others *why* it is true." This puzzle also supports SLO 3 (once again, the properties of the isosceles right triangle, but also here the definition of an exterior angle.)

Pentominoes

Another fixture of recreational mathematics is the *polyomino*, a generalization of the domino. A polyomino is a figure created by joining a given number of unit squares edge-to-edge. They were named and explored by Solomon W. Golomb [14]. Pentominoes are the particular case for five squares. As shown in figure 14, there are twelve distinct shapes, which makes them more manageable than the 35 hexominoes. Thus, there is a vast literature about pentominoes.[13] Their usefulness in the classroom means that plastic pentominoes are widely available in educational catalogs.[14]

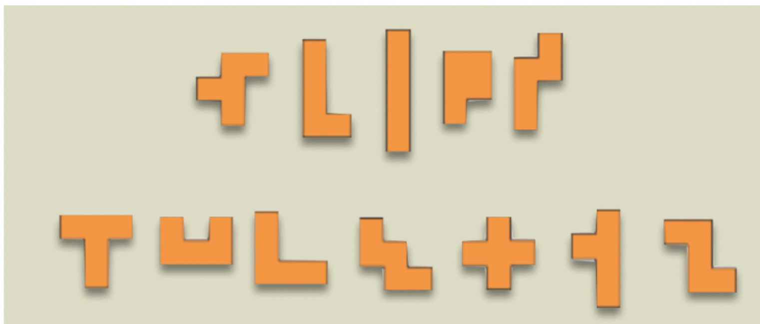


Figure 14.

The vague resemblance of the pentominoes with letters of the alphabet has led to each piece having a name: FLIP'N, TUVWXYZ.

The initial challenge is to find them all, which students can do on grid paper. This naturally leads to a discussion of what constitutes a different find. It is not unusual for students to "find" the same figure more than once, in a different orientation, as in Figure 15.

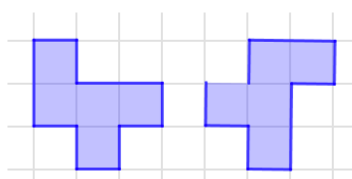


Figure 15.

For the purpose of further work with these shapes, we agree on a convention: we will count congruent shapes as being the same whether rotated or reflected.

This is by no means an obvious choice. For example, in the video game Tetris, which is based on tetrominoes (four squares joined edge-to-edge) the convention is to consider rotated pieces

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[13] See for example [8] (chapter 19,) [9] (chapter 13,) and [7] (chapters 3 and 6.)

[14] I brought pentominoes into many K-12 classrooms in puzzle books which are now available as free downloads at mathed.page/puzzles/pento-books. You can find more pentomino activities here: mathed.page/puzzles/pento-labs/

as equivalent, but reflected pieces as different. Another way to put it: Tetris pieces are *one-sided* — they cannot leave the plane in order to be flipped over. This comparison can yield a worthwhile discussion of definitions (SLO 5,) as well as congruence, reflection, and rotation (SLO 10.) SLO 5 is “Understand the role of definitions in mathematical discourse.” It states that students should “compare and contrast definitions that refer to different properties.” SLO 10 points out that congruence can be defined “in terms of rigid motions: two figures, A and B, are considered congruent if, and only if, there exists a sequence of rigid motions, *r*, that superimposes figure A onto figure B.”

The best-known pentomino puzzle is the challenge to arrange all twelve pieces into a rectangle. It is probably not a good idea to do this in class, as it is time-consuming and the curricular benefits are few[15]. For the purpose of a GeT course, the main use of pentominoes is probably in the exploration of **pentomino blowups**. The idea is to **create a scaled version of each, and try to cover it with pentominoes from one set**. Naturally, “from one set” implies that pieces cannot be re-used within a single puzzle.[16]

If the dimensions are doubled, ten of the resulting puzzles can be solved. Figure 16 shows an example.

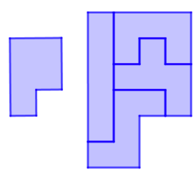

Figure 16.

[15] However, in middle school, a worthwhile exploration is “**what rectangles of any size can be covered with pentominoes from one set?**” (i.e., without using any pentomino shape more than once.) This initiates a conversation which can lead to this conclusion: the area must be a multiple of 5, and therefore one or the other side of the rectangle (or both) must be a multiple of 5. This condition is necessary, but not sufficient. A 2x5 rectangle cannot be solved: if you place a pentomino in it, you need another copy of the same pentomino to cover the rectangle, and that would violate the assumption that you must use pentominoes from a single set.

[16] Certainly other versions of this challenge are possible, for example: “Use the P pentomino repeatedly, and only the P pentomino, to cover scaled pentominoes.”

The doubled X and V cannot be covered with distinct pentominoes. For example, once you place one pentomino in an apparently promising position in the doubled X, you realize the only way forward is to reuse the same piece, which of course is no longer available! (See figure 17.)

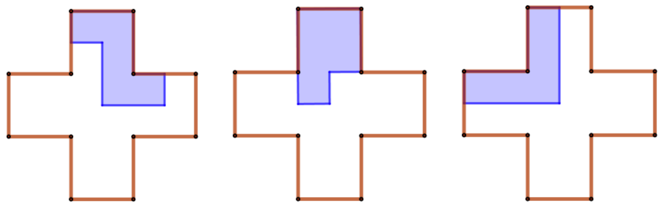

Figure 17.

If the dimensions are tripled, all twelve puzzles are possible. Figure 18 shows a "triplicated" P pentomino.

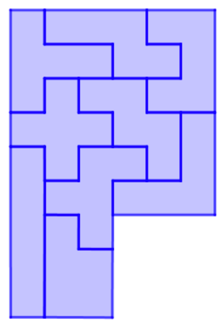

Figure 18.

The fact that you need, respectively, four or nine pieces to solve the puzzles is an instance of the theorem which states that if the ratio of similarity is k, then the ratio of areas is k^2 (SLO 3.) Solving these puzzles does not constitute a proof, but it does beg for one.[17]

High school students enjoy coloring their puzzle solutions. Those can be shared on a classroom bulletin board, where they serve as a reminder of the theorem.

If one does not have access to physical pieces, pentomino puzzles can be solved online on several sites.[18]

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[17] One proof for polygons is based on the fact that any polygon can be dissected into triangles. See for example [3] pp. 391-393.

[18] For example mathed.page/puzzles/pentominoes

Supertangrams

This can be taken further by looking at *supertangram* blowups. Supertangrams (also known in recreational mathematics as *tetraboloes* or *tetratans*) are the figures created by joining four isosceles right triangles (each one half of a unit square) edge-to-edge.[19]

Once again, the initial activity is the discovery of the 14 possible shapes, presumably by experimenting on grid paper. (This is a classic classroom activity known as "the four triangles" in elementary schools, though in that context it is carried out with the help of paper triangles.) Adjoining triangles must share congruent sides. Arrangements such as the ones shown in figure 19 are not allowed, since they violate that rule.

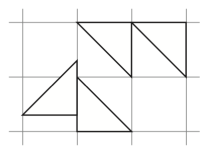

Figure 19.

Once again, there is a necessary discussion of shapes which are reflections or rotations of each other, and thus are congruent (SLOs 5 and 10.) The two shapes in Figure 20, for example, are reflections of each other, so they are congruent and are really the same shape for puzzle purposes.

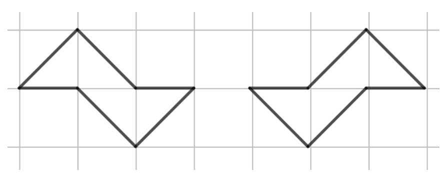

Figure 20.

Figure 21 shows the complete set.

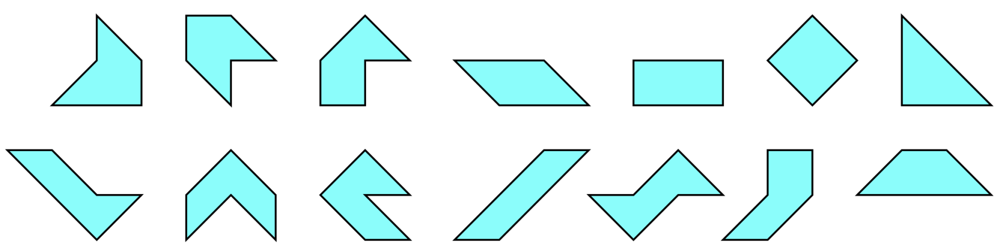

Figure 21.

[19] I came across them in [9] (chapter 11.)

In some cases it is possible to **create puzzles by scaling a supertangram so that it can be covered with two, four, eight, or nine pieces. What is the scaling factor in each case?** Figure 22 shows an example.

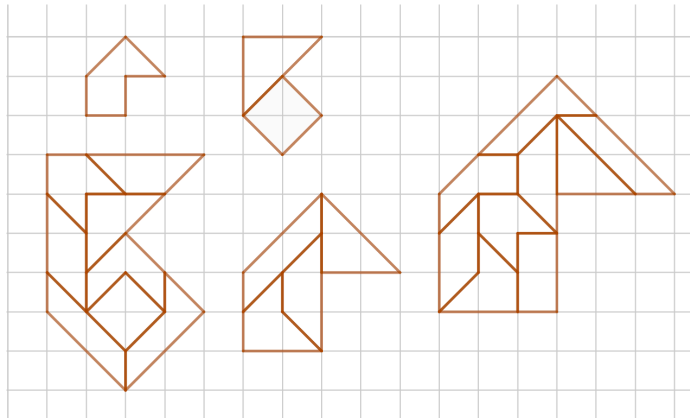

Figure 22.

Once again, this addresses SLO 3: the isosceles right triangle (as in the earlier work on tangrams,) similar figures and the ratio of area (as in the work on pentominoes.) However, this puzzle goes further: as in the case of pentominoes, if you double the dimensions, you multiply the area by 4. But what happened to the dimensions of the supertangram figure if it is the area that was doubled? Students should be able to figure out that the scaling factor was $\sqrt{2}$. Scaling pentominoes involves only whole numbers and ratios of whole numbers. Extending this to supertangrams is an opportunity for students to work with radicals, another standard topic in high school geometry.

Students very much enjoy solving the two larger puzzles in figure 22. However, in the absence of physical pieces, they are probably too challenging — certainly they would be too time-consuming to pursue in the classroom on grid paper. Still, one can get at the underlying mathematics by sharing the image with the class, and discussing the ratios of perimeters between any two figures in that representation.

Students who develop an interest in supertangrams can be challenged to create a figure like Figure 22 for other supertangrams. It turns out that six of the 14 yield puzzles which can be solved at all four scales: $\sqrt{2}$, 2, $2\sqrt{2}$, and 3 — requiring respectively 2, 4, 8, and 9 pieces. As far as I know, finding which pieces those are can only be done by brute force experimentation.[20]

[20] For a supertangram classroom unit, see mathed.page/puzzles/supertangrams. Many solutions to the scaling problems can be found in the *Supertangram Activities* books, which can be downloaded here: mathed.page/puzzles/supertangrams/st-books.html

Rep-Tiles

The scaling puzzles involving pentominoes and supertangrams suggest a more general question: can we tile a given polygon with scaled copies of itself? Figure 23 shows that the P pentomino has this property.

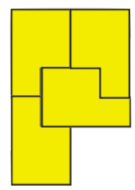

Figure 23.

Such shapes are known as **rep-tiles,**[21] and they are a fine topic for exploration in a GeT course. In particular, this prompt can trigger a worthwhile curricular search.

> **Find a triangle that can be tiled by 2 scaled copies of itself.**
> **Repeat for 3, 4, 5, 8, 9, and 10 copies.**
> **Are other numbers possible?**

Figure 24 shows some solutions.

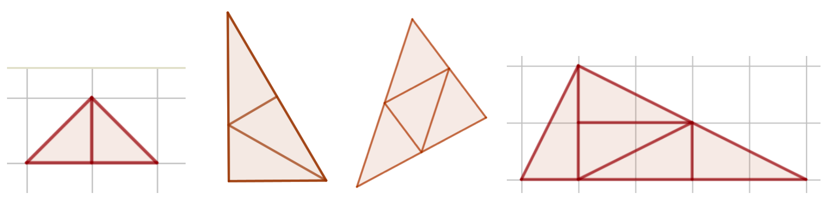

Figure 24.

Any triangle can be tiled with 4 or 9 scaled copies of itself. The isosceles right triangle works for 2, and the half-equilateral (30-60-90°) triangle works for 3. The 1, 2, $\sqrt{5}$ right triangle works for 5.[22]

This quest once again supports SLO 3, by bringing into play essential understandings which are present in any geometry standards: congruent and similar figures, special right triangles, and the Pythagorean theorem. For example, to split a triangle into five congruent pieces, the scaling factor must be $\sqrt{5}$, which suggests the right triangle with legs 1 and 2, since its

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[21] They were named by Solomon W. Golomb. See [11], chapter 19.

[22] Using its rep-tile decomposition, John Conway showed that the 1, 2, $\sqrt{5}$ triangle can tile the plane nonperiodically, with tiles appearing in infinitely many orientations. This could yield an interesting side trip for a GeT course [19].

hypotenuse is $\sqrt{5}$. This, in turn, suggests a question: **Can all right triangles whose leg lengths are whole numbers be rep-tiled?**[23]

The Geoboard

So far, I have shared tiling and dissection puzzles of various types. I will end with something different. The geoboard is an array of pegs, which represent a square lattice of points on the plane, as shown in Figure 25. Students create figures by placing rubber bands on the pegs. In secondary school, it is a rich environment for exploration of topics such as area, slope, the Pythagorean theorem, Pick's formula, and more.

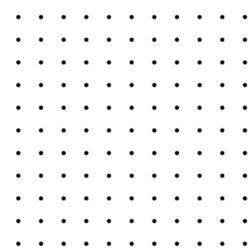

Figure 25.

The geoboard is an example of a *microworld*, in the sense pioneered by Seymour Papert. A microworld is a "subset of reality,"[24] a constrained environment in which a learner can explore powerful ideas which apply to the broader world. In this case, working on a lattice is helpful preparation for more general work on the plane. For a GeT course, I suggest these puzzles for a standard 11 by 11 pegs geoboard.[25]

1. Make geoboard squares of every possible size, and find their areas.[26]

Initially, students find squares whose sides are parallel to the board's edge, then at a 45° angle, before realizing that any small enough geoboard segment can be the side of a square. The challenge in finding the "tilted" squares is to confirm that consecutive sides are indeed perpendicular and congruent. For GeT students, this can be an opportunity to use the "opposite reciprocal slope" property of perpendiculars, and the SAS congruence property of triangles. With high school students I prefer more informal ways to verify these facts at this stage of the unit: I suggest they use a piece of paper's corner to check the angles, and a ruler to measure the sides, saving the formal proofs for later.

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[23] Snover et al [18] provide an answer as part of their full analysis of triangle rep-tiles.

[24] Papert ([16] p. 204) was referring to software tools, but the basic idea does apply to the geoboard, and in fact to various manipulative environments for math education.

[25] 5 by 5 pegs geoboards are more common in elementary schools, but they are not well-suited for the work I suggest here.

[26] This is a summary of *Geometry Labs* 8.5, 9.2, 9.3, 9.4 (pp.115-127)

To find the area of such a tilted square, students can subtract four triangles from the larger square. Figure 26 shows an example.

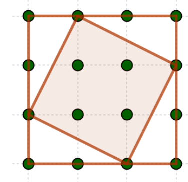
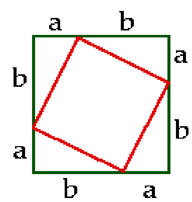


Figure 26.

The larger square has area 9, each triangle has area 1, and thus the inside square has area 5. The figure, generalized, leads to a standard proof of the Pythagorean theorem, as shown in figure 27. (For the proof to be rigorous, it is important to prove that the inner figure is indeed a square. This can be deduced from the congruence of the four triangles and the fact that their acute angles are complementary.)



outer square: $(a+b)^2$

each triangle: $\dfrac{a \cdot b}{2}$

inner square: $(a+b)^2 - 2ab = a^2 + b^2$

Figure 27.

Having carried out this calculation repeatedly with actual numbers makes the proof much more accessible to high school students. This is an important pedagogical insight: abstraction is built on a concrete foundation. SLO 3 states: "It is not enough for preservice teachers to know the content; these students must also gain specialized pedagogical knowledge to teach effectively." Merely stating this proof in a lecture is not likely to reach a wide range of students. It is vastly more effective to present this after students have engaged with many specific cases. In fact, at that point, it is possible for the students themselves to make the generalization.

Work on the geoboard can also help students get some insight about square roots. Arranging four copies of the square from Figure 26 as shown in Figure 28 suggests that $\sqrt{20} = 2\sqrt{5}$. This provides a geometric justification for the standard approach to simplifying radicals.

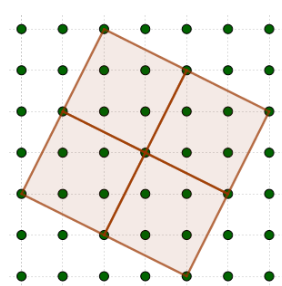


Figure 28.

2. Make triangles of area 15, such that no side is parallel to the edge of the board.

This is surprisingly difficult to solve by trial and error, so students will need some other strategies. One way to do it is to find ways to factor 30 using available geoboard distances, such as $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{10}$, and use them to choose the base and height of the triangles appropriately. In Figure 29, the base is $3\sqrt{5}$ and the height is $2\sqrt{5}$.

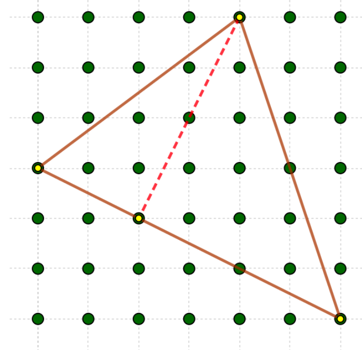


Figure 29.

Additional solutions can be obtained from this one by moving one of the vertices in a line parallel to the opposite side, as that does not affect base or height. For example, replacing vertex A with A', as in figure 30, yields another triangle with base $3\sqrt{5}$ and height $2\sqrt{5}$, so the area is unchanged.

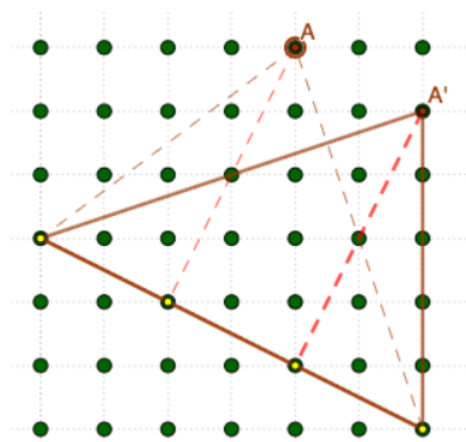


Figure 30.

Naturally, the target area need not be 15, as long as it is a number which can be factored in such a way as to solve the puzzle![27]

[27] This puzzle begs the somewhat less curricular question: what areas are possible / impossible for any geoboard shape? As it turns out, all geoboard areas are multiples of ½. This is a consequence of Pick's formula. See *Geometry Labs* 8.6 (pp. 116-117) for an activity leading to the formula, and mathed.page/ geometry-labs/pick for a proof. Conversely, every multiple of ½ can be the area of a geoboard shape — this can be shown by mathematical induction.

3. **Make isosceles triangles**, with the following constraint for each: it cannot be a right triangle, and the base cannot be parallel or at a 45° angle to the board's edge.[28]

Again, this is far from obvious. One winning strategy is to start with a base whose midpoint is a lattice point, and to place the apex on its perpendicular bisector, as in figure 31.

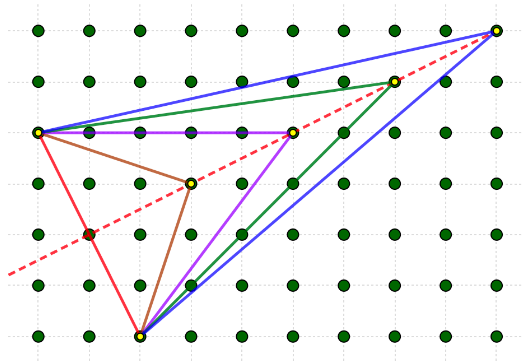


Figure 31.

Geoboard puzzles do not absolutely require a physical geoboard: all can be done online,[29] or on a paper representation of the board — dot paper is preferable,[30] but grid paper will do. They address SLO 3: area of squares and triangles, the Pythagorean theorem, working with radicals, the perpendicular bisector, and perhaps more (depending on the strategies implemented by the students.)[31]

Conclusion

The puzzles presented in this chapter are instances of a puzzle-rich pedagogy.[32] It is my hope that the approach they embody helps to promote the idea that puzzles can and should enrich precollege math education. In my experience, puzzles are intrinsically more interesting to more students than the definition-theorem-proof routine, but they should not be counterposed to it. Their use provides a foundation and a context for the development of geometric concepts.

Still, it would be a mistake to think that the puzzles can do the teaching. Many, perhaps most, GeT students will not spontaneously see the math which is embedded in some of these puzzles: what they will see is limited by what they already know and understand. It is the role

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[28] One version of this problem, with answers in the back of the book, is at the end of Lab 9.2 in *Geometry Labs* (pp. 123-124.)

[29] See for example mathed.page/geoboard/geoboard

[30] mathed.page/geoboard/dot-papers.pdf

[31] For much more about the geoboard, see mathed.page/geoboard

[32] For a broader discussion of puzzles in math education, see mathed.page/teaching/puzzles.html

of the instructor to use the puzzles as a springboard for reflection, discussion, and writing — and to make the connections with the curricular goals of the GeT course.

Acknowledgements

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