

Infinity: Teacher's Guide

"Infinity" is a course I have been teaching at the Urban School of San Francisco, every other year since 1991. I offer this Teacher's Guide to other teachers who may be interested in teaching a version of the course, or some parts of it. (See www.MathEducationPage.org/rights.html for the conditions under which you have my permission to use these materials.)

The worksheets do not add up to a full textbook. You will need to complement them with the suggested readings, or equivalent, and with teacher explanations. See also www.MathEducationPage.org/infinity, and www.edWeb.net/escape for additional materials.

The course is intended for a teacher who is already familiar with the concepts covered. These materials should help such a teacher bring deep and powerful ideas, usually not tackled until college, to an advanced high school class. The approach of the worksheets is developmentally appropriate to high school, and does not attempt to present the material in the style of college math classes.

The most common mistake in attempting to teach this material is to underestimate its difficulty, even for a strong high school student. It is crucial to provide adequate scaffolding between standard high school fare and the substantially more abstract subject matter of this course. This is what I prioritize in this packet, and in the lessons I link to on the Web site. It is also important to give enough time for the tougher ideas to penetrate and to offer opportunities to think about them repeatedly, in different forms and representations.

The course consists of four units:
- Unit 1: Different-sized Infinities
- Unit 2: Chaos
- Unit 3: Mathematical Induction
- Unit 4: Fractals

I listed them in the order I teach them, but the units are largely independent of each other, and one could probably teach them in a different order. Moreover, it is entirely possible for a teacher to incorporate one or more of the units in a precalculus class or other math elective, without needing to do the whole thing.

Two common threads that tie the units together (besides of course infinity) are formal proof in units 1 and 3, and computer-based mathematics in units 2 and 4.

On the next page: the "welcome" sheet I hand out on the first day of the class. Aside from an overview of the topics, it includes a description of what will be expected of students. Note that it lists a substantial list of "algebra review" topics. These are mostly not represented in these materials, since I mostly use exercises from assorted textbooks – I am sure you can find your own way to do this (necessary and important!) review work.

Infinity

Welcome to Infinity!

Here is the course description:

> We open the course with a discussion of ancient and modern paradoxes involving infinity, and students are introduced to Georg Cantor's discoveries. Students are expected to understand and be able to reproduce his key arguments about infinite sets, including some subtle proofs by contradiction.
>
> We explore various sequences and series, including especially the Fibonacci sequence. Much of this work requires the use of proof by mathematical induction, which students are expected to master.
>
> Students use computers to explore dynamical systems, and thus are introduced to chaos, a new multi-disciplinary mathematical science. In this context, we pay special attention to the iteration of the logistic function, a mathematical model of population dynamics. Students write a report about the underlying mathematics.
>
> We explore fractal geometry through students programming their own images and calculating fractal dimensions. The course ends with an introduction to dynamical systems in the complex number plane, including the study of Julia sets and of the Mandelbrot set, one of the most spectacular discoveries of contemporary mathematics.
>
> We make connections to literature, to the history of math, to science, and to other branches of math, including some reading and some review of algebra throughout the course. Iteration and recursion are concepts from mathematics and computer science that tie many of these topics together, and allow us to get a mathematical handle on infinity.

Expect a difficult course and a lot of work, but it will be deep, exciting, and beautiful.

The algebra review mentioned above includes:

prime numbers,

algebraic fractions,

similarity and proportional reasoning,

sequences and series,

iteration of linear functions,

logarithms,

complex numbers,

…and perhaps more.

The reading comes from:

Galileo,

Jorge Luis Borges,

Douglas Hofstadter,

Martin Gardner,

Lewis Carroll,

James Gleick,

Scientific American,

… and perhaps more.

Unit 1: Different-Sized Infinities!
Set Theory and Proof by Contradiction

Teacher Notes

Start the unit by having students read aloud Galileo's discussion on infinity from the *Dialogue on Two New Sciences* (1638). It is excerpted in Sherman Stein's *Mathematics: The Man-Made Universe* (2nd edition, Freeman, 1962), pp. 314-315, and can be downloaded from my Web site
http://www.MathEducationPage.org/infinity/galileo.html
The ensuing discussion can help set up many of the key questions in this unit.

Other readings (required):

"Aleph-null and Aleph-one", Martin Gardner, from *Mathematical Carnival* (Knopf, 1975)

"Georg Cantor", Joseph W. Dauben, in *Scientific American,* June 1983, pp. 122-131

Also (optional):

"The Book of Sand", Jorge Luis Borges, from *The Book of Sand* (Dutton, 1977)

"Hilbert's Hotel", Carol Lay, < http://www.buzzle.com/showImage.asp?image=272>

Not included in the worksheets: discussion about paradoxes involving infinity. Some reading on this:

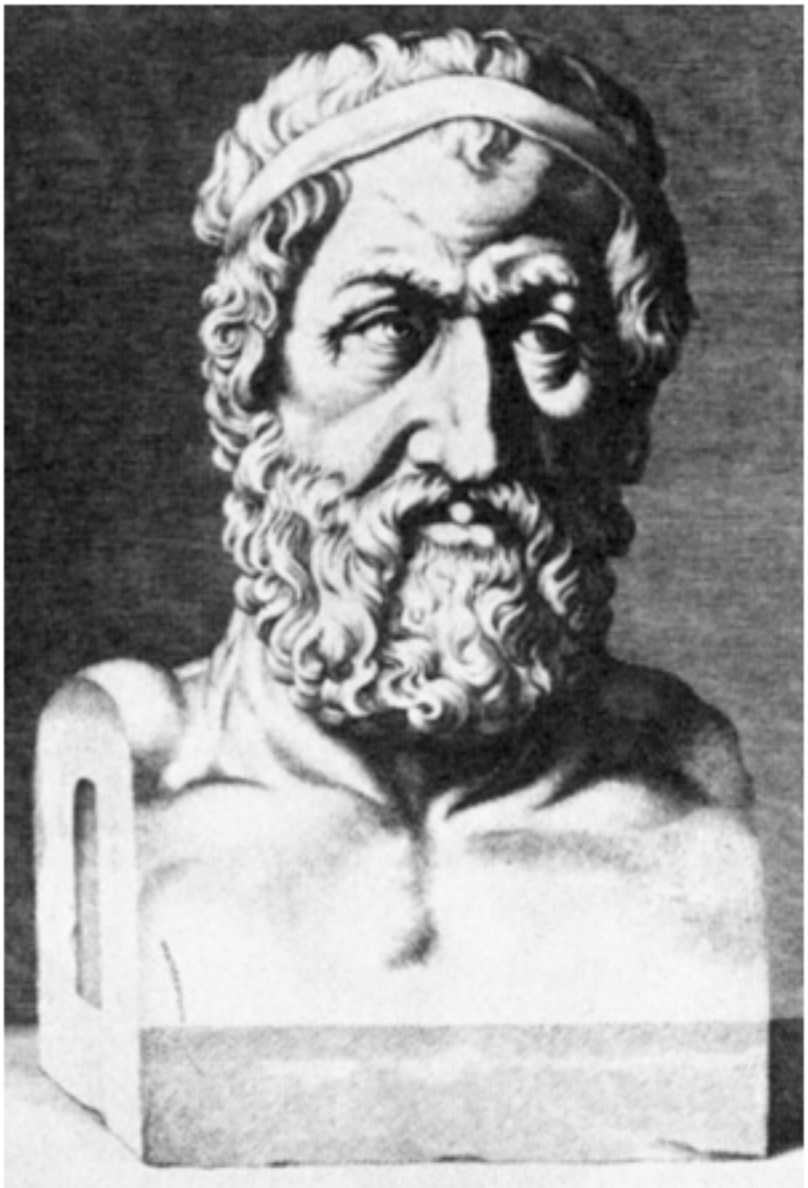
Aha! Gotcha, Martin Gardner, pp. 16-17, 143-144

Prime Numbers:

The purpose of this activity is two-fold: on the one hand, it introduces proof by contradiction, a central tool in mathematics and in the discussion of infinity; on the other hand, it addresses one question about infinity rigorously, in a context that is reasonably familiar to students.

The argument here assumes that every natural number has a prime factor, which students readily accept. One way to explain that is that every number must have a smallest factor (greater than 1), which must be prime. If it were not so, it would have a factor (other than 1 and itself) that would have to be a factor of the original number, which would contradict our assumption that this was the smallest factor.

The approach in this lesson is largely based Sherman Stein's *Mathematics: The Man-Made Universe,* Chapter 4.



Zeno (ca. 490-430 BC?)

Georg Cantor

1845-1918

Unit 2: Chaos!
Dynamical Systems

Teacher Notes

I usually start this unit by having students read the Prologue to James Gleick's *Chaos*.

Iterating Linear Functions

There is a prerequisite unit to this lesson (and this whole unit,) also titled "Iterating Linear Functions". It is linked to on the Infinity Web page on my site, and is suitable for students in Algebra 2. This lesson reviews that unit, and takes it one step further towards an explicit formula for the n^{th} term.

In addition to reviewing that foundational work, this worksheet provides a good context to discuss the different types of orbits (see the lesson: "Orbit Analysis"), and to introduce graphical iteration.

Graphical Iteration

Students should first do graphical iteration (as described for example in Douglas Hofstadter's *Metamagical Themas,* Chapter 16) by hand, to understand how it works. Later, it is necessary to use electronic support for this. I use a homemade program in the Boxer programming environment for this, but a more accessible tool is a Texas Instruments graphing calculator – most of them support graphical iteration in sequence mode, under the name "web plots".

See also the Fathom files on the Web site if you have access to the Fathom software. They provide a crucial electronic approach to this unit. (If you end up doing this using spreadsheets or some other technology, please send me your files, and I will post them on the site for others to use. Of course, I will acknowledge you.) Note that there are two unnumbered Fathom files with two-dimensional iterations.

Iterating the Logistic Equation

In the same chapter, Hofstadter explains what happens when iterating the function $f(x)=r\cdot x\cdot(1-x)$. (Actually he uses 4λ for r.) You will find a more accessible introduction to the same thing in Gleick's *Chaos*, chapter 3. Both are required readings, though students need not read the whole of Hofstadter's chapter: the crucial part is pp. 364-376.

The worksheet addresses many key ideas that neither author emphasizes, but which are nevertheless important foundational ideas for this work.

Note that $f(f(x)) = -r^3x^4 + 2r^3x^3 - (r^3+r^2)x^2 + r^2x$. (This formula can be used to experiment with a key idea raised in Hofstadter's chapter.)

Building a Bifurcation Graph

The idea here is for students to build their own bifurcation graph, using Fathom. This reinforces the difficult ideas encountered in the reading.

Orbit Analysis of a "Dome" Function

Students apply what they learned so far in this unit to another "dome" function. This is an excellent way to assess their understanding. I use the subsequent sheet **Assessment: Orbit Analysis of a "Dome" Function** to give students feedback on how well they met my expectations.

Iterating a Function of Complex Numbers

It is of course necessary to learn or review complex number arithmetic before this worksheet. Then, as an introduction, start by figuring out the real and complex parts of the recursive formula, and finally have the students build this in Fathom – see the Fathom file **complex-iter.ftm** on the Web Site.

It is interesting to use Fathom to explore and discuss what happens with different seeds for a given c, and it provides a necessary foundation for understanding Julia sets and the Mandelbrot set. (It is not too difficult to find Web sites where students can explore those but doing that without laying the necessary groundwork would not be nearly as meaningful.)

Assessment: Orbit Analysis of a "Dome" Function

Find the maximum value of r that keeps your function inside the unit square for values of x between 0 and 1.

For what value of r do we get extinction? equilibrium? cyclical behavior? chaos?
(for each of these behaviors give a description and examples)

Where are the fixed points?

How do they show up graphically?

Do they attract or repel?

What are the points of bifurcation?

(What does that mean?)

Explain this in terms of the graph of the function f(f(x), etc.

Describe unusual features of your particular function.

Try to find values of r that give orbits for which the period is not a power of 2

Include a short introduction (or conclusion) based on the readings, about how this sort of analysis fits in the recent history of science.

Use language correctly.

Write clearly, and *illustrate*.

Unit 3: Stairway to Infinity!
Proof by Mathematical Induction

Teacher Notes

The Strong Law of Small Numbers

For much of their schooling, our students have been encouraged to look for numerical patterns, and to generalize them. While this is a sound pedagogical practice, as it helps develop number sense and lays a foundation for algebraic thinking, it does reinforce the very non-mathematical misconception that "if it seems true, it must be true". This worksheet works well to challenge the misconception.

Selected answers:

5. breaks down for n=40
6. breaks down for n=127
7. breaks down for n=432
8. breaks down for n=5777

Introduction to Mathematical Induction

In Algebra 2, my students become familiar with the concept of explicit vs. recursive formula for a sequence. I find that this provides a good foundation for introducing the idea of proof by mathematical induction, though of course it is not sufficient.

Do not expect students to get through this worksheet in one sitting!

In many, if not most, textbooks, when proof by mathematical induction is introduced, students are supplied with the conjectures that they are asked to prove. I find that they are much better motivated to write the proofs if they themselves come up with the conjecture. Of course, this is more difficult, and it requires providing much support to the students, including hints. See the visual hint for the sum of cubes on my Web site. I also encourage the students to use Fathom (or a spreadsheet program) to test their conjectures. (See the Fathom files on the Web site for examples.)

The triangular numbers, of course, come up in a multiplicity of disguises: the "handshakes" problem, connecting n vertices with edges, arranging chips into three non-empty piles, number of line segments determined by n points on a line, and so on.

Another version of pentagonal numbers: omit the central dot.

A number-theoretic conjecture that can be proved by mathematical induction, but not following the explicit vs. recursive formula approach laid out on this worksheet: every power of 4 a multiple of 3, plus 1. I usually do this a couple of days into this unit, to clarify that the essence of mathematical induction is not the algorithm presented on this sheet. (Though I have found that for most students, the approach on this sheet is an effective way to get started.)

Fibonacci Conjectures

Prerequisite: I usually introduce Fibonacci numbers (which many students have already encountered) by asking how many ways there are to tile a 2 by n strip with dominoes. This is useful in previewing the domino-based proofs we look a little later. An equivalent problem is “How many ways are there to climb a staircase, if one can take either one or two steps each time?”

The next step is to point out that any two numbers can serve as the seeds of a Fibonacci-style sequence, and ask students to make their own. Then I have them add two such sequences, and note that this creates yet another Fibonacci-style sequence. In particular, when a student hits upon the Lucas numbers (seeds: 1, 3), I tell them that this is what that particular sequence is called. If no one comes up with it, it is usually sufficient to ask for the “simplest” Fibonacci-style sequence for someone to suggest it.

Finally, I have students look for patterns in the Fibonacci and Lucas numbers, as a way to come up with conjectures. Some students find it helpful to use Fathom tables as an environment to generate and test their ideas. If students are stuck, I hand out this worksheet to help them get started.

Of course, the purpose of the activity is to end up proving the conjectures. Assuming two consecutive steps for the formula, adding them, and doing a little algebra, sometimes leads to a proof by mathematical induction. On the other hand, some of the formulas are not so easy to prove, which provides motivation for finding an explicit formula for the Fibonacci and Lucas numbers.

Unit 4: Fractals!
Recursion

Teacher Notes

I do this unit in the Boxer computational environment, using its turtle graphics capabilities. Any full programming language with full access to turtle graphics, the ability to pass parameters to procedures, and support for recursion would presumably work just as well. In particular, this should be possible in most versions of Logo. (Alas, not all!) I understand that one can also do some of this work with Geometer's Sketchpad.

What is exciting about fractals, aside from the mathematics, is the fact that they provide a wide-open opportunity for student creativity, something that is often lacking in math classes.

I have no worksheets on this: I teach it pretty much entirely at the board, and on the computer. I introduce three methods for drawing fractals via turtle graphics: "self-decorating" fractals, where smaller copies of the overall shape are embedded in the image; "replacement" fractals, such as the famous "snowflake", where part of a design is replaced with a more complicated version of it; and "tree" fractals, where the figure branches into smaller versions of itself. Students do a final fractal project where they create fractals by using one or more of these techniques.

Lewis Carroll's "Two-Part Invention" (available on my Web site) is a logical —not visual— fractal, and it makes for interesting philosophical discussions. Martin Gardner cites it in "Infinite Regress", a wonderful chapter in his *Sixth Book of Mathematical Games from Scientific American*. In it, he makes connections between fractals and other cultural manifestations of infinite regress: in art, theater, mathematics, etc.

See the next page for a few end-of-unit exercises and problems.