Iterating Linear Functions

Lessons from *Algebra: Themes, Tools, Concepts*
by Anita Wah and Henri Picciotto

Teacher's Notes

Instant Riches

Core Sequence: 1-12

Suitable for homework: 5-15

Useful for Assessment: 12-15

What this Lesson is About:
- Iterating linear functions
- Solving a real world problem by trial and error and by working backwards
- Using mathematics as a tool for decision-making

Amazing Opportunity!

This problem is difficult enough to challenge all students, but simple enough to allow everyone to participate. A more fully algebraic discussion of the problem is saved for "**More Banking**".

The exploration should lead to some interesting discussions, and will be a good way to make sure everyone understands the big question that this lesson addresses.

Students are likely to use trial and error for #2. #3 should get the point across that the plan is advantageous to some, but not to others. In addition, it supplies students with a new tool: using arrows to represent functions.

Running Out of Money

This section is a generalization of #2, hopefully leading students to seeing the benefits of working backwards for solving this sort of problem. (For example, in #4, it is easy to see that if subtracting $100 from Bea's balance left $0, then there must have been $100 in the account, which means that she must have started with $50.)

If some groups continue to use trial and error, a class discussion of different strategies could be helpful, so that everyone understands the strategy of working backwards. The arrow representation makes it easy to trace one's way backwards. The key is to understand that to travel backwards along an arrow, you have to use the inverse operation.

Gaining and Losing

#9-11 are intended to nudge the students towards a thorough understanding of who gains and loses and why. The report in #12 will give them a chance to show their understanding. However, if the report does not include a mathematical explanation of the problem, insist that such an explanation be present in #9-11.

#13-15 are optional extensions, which you can use to explore this problem in more depth. In either case, Writing Assignment C will be a chance to come back to the problem at a higher level of understanding.

#15 is deliberately vague. It allows students to design plans that are very similar to the one presented in this lesson. But very interesting mathematics can emerge from an open-ended question of this type. For example, a plan that doubles the investment and charges no fee will yield a geometric sequence. A plan where a certain amount is added to the account each month but the fee is a percent of the account behaves yet another way. If you do not want such an open-ended situation, skip the problem, or specify the limits you want your students to respect.

More Banking

Core Sequence: 1-11

Suitable for homework: 1-12

Useful for Assessment: 11, 12

What this Assignment is About:

- Combining functions
- Using variables and function diagrams to analyze a real world problem

In this assignment, we go back to the problem that opened this packet. The difference is that now, students should be able to analyze the problem at a somewhat higher level, with the help of variables and function diagrams.

Iterating a function of the type $y = mx + b$ yields a sequence of numbers that is not that easy to analyze with complete generality. Do not expect your students to completely understand all the ramifications of this problem.

For your own information: if the formula was simply of the form $y = x + b$, we would have an arithmetic sequence; if the formula was of the form $y = mx$, we would have a geometric sequence; finally, the form $y = mx + b$ with $-1 < m < 1$ yields a sequence that converges towards a limit.

Iterating Linear Functions

Core Sequence: 1-14

Suitable for homework: 5-13

Useful for Assessment: 13

What this Lesson is About:

- Iterating linear functions
- Recurrence equations
- Convergence
- Fixed points

The ideas in this lesson were previewed in “**Instant Riches**” and “**More Banking**”. In more advanced classes, students may study an extension of these ideas to non-linear functions. This is the essential core of the science of chaos and dynamical systems.

This lesson was inspired by an article in the *Mathematics Teacher*, vol 83, #9, December 1990: “The difference equation $x_n = a\ x_{n-1} + b$”, by Lawrence E. Spence.

A hands-on experiment to bring this topic to life in the classroom with water and food coloring is described in the *Mathematics Teacher*, vol 85, #2, February 1992: “Drugs and Pollution in the Algebra Class”, by James T. Sandefur. In the experiment, an empty glass container represents the body, a quart of water represents the blood in the body, and 16 ml of food coloring represents 16 mg of medicine. The elimination of 25% of the medicine in a 4-hour period is modeled by removing one cup of the mixture, and replacing it with a cup of clear water.

Tree Harvesting

These three problems introduce the basic idea of the lesson. Calculators are essential, and programmable calculators or computer spreadsheets are particularly convenient.

Drugs / Savings

Function diagrams help students visualize what happens in the long run. In particular, students should see a major difference between the drugs and the savings problems, in that the in-out lines converge in the first case, and diverge in the other.

Note that in the case of the drug problem, if one uses the equation $y = .6(x + 10)$, the numbers will converge to a different number. This is because in this case, the formula describes the amount of the drug in the body just before taking the next dose. The equation in the text reflects the amount in the body just after taking a dose.

#14 should remind students about the analysis of the focus of function diagrams for linear functions (from Chapter 8, Lesson 2.) We deliberately ignore the cases where $m \leq 0$ for the purposes of this lesson. They can be analyzed in roughly the same way.

Analyzing the Sequences

In the long run, the sequence of inputs and outputs gradually gets closer and closer to, or farther and farther from, the function's fixed point, depending on whether $m < 1$ or $m \geq 1$.

It is interesting that the initial input does not affect the outcome all that much. The only thing that matters is whether the initial input is above or below the fixed point.

This section pulls together everything that was discovered in this lesson.

Asthma

This is an additional problem, which can be used for review or assessment.