Chapter 6 Combining Operations

This chapter is a transition into the more advanced uses of the Lab Gear. It extends, concludes, reviews, and combines the concepts and skills that were introduced in previous chapters: parentheses, the distributive law, rectangles, inequalities, perimeter, and surface area.

New Words and Concepts

This chapter introduces the following concepts:

Order of operations
Work with the three familiar **identities**
Long division of polynomials

The first two of these are extremely important. The third topic (long division) is rather complicated, and you may decide that it is not worth the trouble, especially since students will not have a use for polynomial division until second year algebra or even precalculus. The Lab Gear long division algorithm does not parallel every step of the traditional algorithm, but it is an interesting and clever application of many of the techniques that have been learned so far. It requires using both the corner piece and the workmat.

Teaching Tips

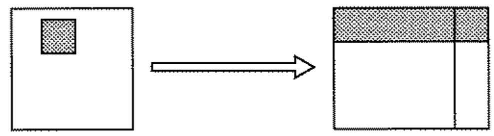
Lessons 1 through 4 pull together the knowledge that has been accumulated in the first five chapters, and prepares the students to put it all to use in the final three chapters. Take the time to make sure the students can verbalize the key ideas. This would be a good time for a "Lab Gear midterm."

Lesson Notes

- **Lesson 1,** Order of Operations, page 72: These exercises do not involve the blocks. They are challenging, and students should be encouraged to work together on them.
- **Lesson 2,** The Distributive Law, page 74: This is the first formal statement of the distributive rule. All the work that has been done until now should have laid the basis for a solid understanding of the law.
- **Lesson 3,** Parentheses Review, page 75: This lesson should help the students consolidate their understanding of parentheses, and it should help you evaluate whether they have mastered the concepts and techniques.
- **Lesson 4,** Three Identities, page 76: Because the students have seen many "Make a Rectangle" and "Make a Square" problems, this lesson should present no conceptual difficulty. See it as an opportunity to start learning to recognize the identities, which will be useful in further work in algebra—factoring, solving equations, completing the square, rationalizing denominators, and so on.
- **Lesson 5,** Long Division With the Lab Gear, page 77 and **Lesson 6**, Long Division Without the Lab Gear, page 80: These lessons are optional.

Exploration 1 Make a Rectangle

As you can see in this figure, it is possible to make an uncovered rectangle from the blocks representing $x^2 - 1$ by adding zero in the form of x and $-x$.



1. Write an equation relating length, width, and area of the uncovered rectangle.

Use the same method to make an uncovered rectangle for these sets of blocks. In each case, write an equation.

2. $y^2 - 1$
3. $25 - x^2$
4. $y^2 - 25$
5. $y^2 - x^2$
6. $y^2 - 9$
7. $4x^2 - 4$

Lesson 1

Order of Operations

Look at the expression, 1 + 2 (3 + 4). Mathematicians agree that it is equal to 15.

1. In what order would you do the calculations to get 15?
2. The expression is not equal to 21, but explain why someone might think it is.
3. The expression is not equal to 11, but explain why someone might think it is.

To prevent the same expression from having several different meanings, mathematicians have agreed on a set of rules to read complicated expressions. This set of rules is called the **order of operations**. It is used not only in mathematics, but also in computer programming languages and in scientific calculators. The rules require that you perform operations in a specific order.

- Exponents and roots first
- Multiplications and divisions next
- Additions and subtractions last

Aside from that, proceed from left to right, and pay attention to grouping symbols such as parentheses. Parentheses are used to override the order of operations rules.

Look at this example. $1 + 2 \cdot 3 + 4 = 1 + 6 + 4 = 11$
This equation is true because multiplications are performed before additions.

However, the addition represented by the second plus sign would be performed before anything else if the operation 3 + 4 was enclosed in parentheses.

$$1 + 2 \cdot (3 + 4) = 1 + 2 \cdot 7 = 15$$

If you have parentheses within parentheses, work from the inside out.

Radical signs and fraction bars also create groups. Study this example.

$\dfrac{\sqrt{1+8\cdot 10}+9}{1-(2+3)}-4=$	work within the radical and the parentheses
$\dfrac{\sqrt{1+80}+9}{1-5}-4=$	work within the radical and the denominator
$\dfrac{\sqrt{81}+9}{-4}-4=$	calculate the radical
$\dfrac{9+9}{-4}-4=$	calculate the numerator
$\dfrac{18}{-4}-4=$	do the division
$-4.5-4=$	do the subtraction
-8.5	

Keeping in mind the order of operations, insert as many pairs of parentheses as needed to make these equations true.

4. $4\cdot 2+3=20$
5. $\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$
6. $5\cdot 3-2+6=35$
7. $3^2+2\cdot 7-4=33$
8. $\frac{1}{3}\cdot 6+4\cdot\frac{2}{6}-\frac{1}{3}=\frac{7}{3}$
9. $1-2\cdot 2+5\cdot 6=-42$
10. $4+6\cdot 2\cdot 5-3=40$
11. $3+1\cdot 7-2^2\cdot 9-7=24$
12. $2\cdot 8\cdot\frac{1}{4}+\frac{2}{3}\cdot 2-\frac{1}{2}=22$

Insert radical signs and/or parentheses to make these equations true.

13. $\dfrac{5\cdot 12+4}{6-2}=5$
14. $\dfrac{2+6\cdot 3^2}{2}=6$
15. $3\cdot 2+4\cdot 2^3=12$

☑ Self-check

4. $4\cdot(2+3)=20$
10. $(4+6)\cdot 2\cdot(5-3)=40$
14. $\sqrt{\dfrac{(2+6)\cdot 3^2}{2}}=6$

The Distributive Law

1. Use the corner piece to work out this product. $(x + y + 1)(x + 5)$
   - Remember that every block on the left must be multiplied by every block across the top. Sketch your result.
2. Use the corner piece to work out this product. $(x + y - 1)(x + 2)$
   - Remember to cancel matching upstairs and downstairs blocks at the end. Write the product.
3. Use the corner piece to work out this product. $(x + y - 5)(-x + y + 1)$
   - Remember to start by multiplying the downstairs blocks with each other.
   - Then, multiply the upstairs blocks with each other (careful!).
   - Next, multiply upstairs times downstairs and downstairs times upstairs. If you did it correctly, the uncovered area should be a rectangle of the right dimensions.
   - Finally, cancel matching upstairs and downstairs blocks. Write the product.

To do multiplications like the above without the Lab Gear, you have to remember the **distributive law**, which states that you must *multiply each term in the first polynomial by each term in the second*. Then you can simplify by combining like terms.

4. Rework problems 1-3 without the Lab Gear. For each one, make sure you get the same answer as before. If not, find your mistake!

Do these multiplications by algebra first, then check your answers with the Lab Gear.

5. $(2x + 3)(3x - 2)$
6. $(-5 + 2x)(x + y - 3)$
7. $(y - 3)(y + 4)$
8. $(2x + y - 5)(y - x + 4)$
9. $(3x - 4)(x + 5 - y)$
10. $(2y - 1 - x)(2x + y - 3)$
11. $(x + y + 1)(2x + y - 1)$
12. $(2x + y - 5)(2x + y + 1)$
13. $(-x + y + 5)(x + y - 5)$

☑ Self-check

1. $x^2 + xy + 6x + 5y + 5$
2. $x^2 + xy + x + 2y - 2$
3. $-x^2 + y^2 + 6x - 4y - 5$

Parentheses Review

Write a summary about the role of parentheses in algebra. Use sketches of the Lab Gear to explain the various rules that you describe. Be sure to cover these four points.

- Parentheses and order of operations
- Removing parentheses that are preceded by a plus
- Removing parentheses that are preceded by a minus
- Removing parentheses that are preceded or followed by a multiplication sign

Exploration 2 Always, Sometimes, or Never?

Always, sometimes, or never true? is an important question in algebra.

1. Make up several algebra statements that are always true.
2. Make up several algebra statements that are sometimes true.
3. Make up several algebra statements that are never true.
4. Discuss your answers with your classmates.

Exploration 3 Positive or Negative?

For each quantity below, indicate with P, N or 0, if it can be positive, negative, or zero. In problems 4, 5, and 6, try replacing with the values –3, 0, 3.

1. 4^3
2. -4^3
3. $(-4)^3$
4. $2y^3$
5. $-2y^3$
6. $(-2y)^3$
7. Which of the expressions in problems 1, 2, and 3 are equal to each other? Explain.
8. Which of the expressions in problems 4, 5, and 6 are equal to each other? Explain.
9. Compare what you discovered about the signs of cubes to what you know about the signs of squares.

Three Identities

Use the corner piece to find these products.

1. $(x + 3)^2$
2. $(x + 5)^2$
3. $(x + y)^2$
4. Explain how to find the square of a sum without the Lab Gear.
5. True or False: The square of a sum is equal to the sum of the squares. Explain, using a sketch of problem 3.

Use the corner piece to find these products.

6. $(y - 2)(y + 2)$
7. $(y - 5)(y + 5)$
8. $(y - x)(y + x)$
9. What pattern did you discover in problems 6-8?

Use the corner piece to find these products.

10. $(y - 3)^2$
11. $(y - 5)^2$
12. $(y - x)^2$
13. Explain how to find the square of a difference.
14. True or False: The square of a difference is equal to the difference of the squares. Explain, using a sketch of problem 12.

As you know, **identities** are equations that are *always true.* The three that are shown in problems 3, 8, and 12 are especially important and useful. For example, using the identity in problem 12:

$$(2x - 5)^2 = (2x)^2 - 2\,(2x)(5) + 5^2 = 4x^2 - 20x + 25$$

Do these problems by using one of the identities, then check your answers with the Lab Gear.

15. $(2x - 3)^2$
16. $(2y + x)^2$
17. $(4y - 1)(4y + 1)$
18. $(3x + 4)(3x - 4)$
19. $(3y + 5)^2$
20. $(5x - 2y)^2$

Do you think there is a pattern for the square of trinomials? Use the corner piece to experiment with these problems.

21. $(x + y + 2)^2$
22. $(x + y - 5)^2$
23. Describe the pattern you discovered in the two problems above.
24. What is $(a - b + c)^2$ equal to? Use the pattern you discovered, then check your answer by using the distributive rule very carefully.

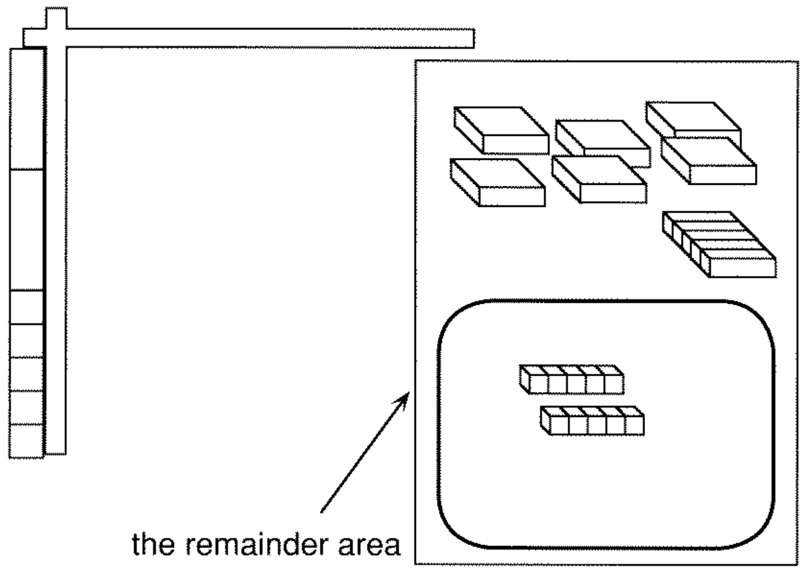
Long Division With the Lab Gear

As you know, it is sometimes possible to divide polynomials with the corner piece. Some divisions are fairly easy to work out by trial and error. Others are harder. Still others cannot be solved with the blocks. The following method will work whenever the division can be modeled with the Lab Gear. It requires one side of the workmat, which will be called the remainder area, as well as the corner piece.

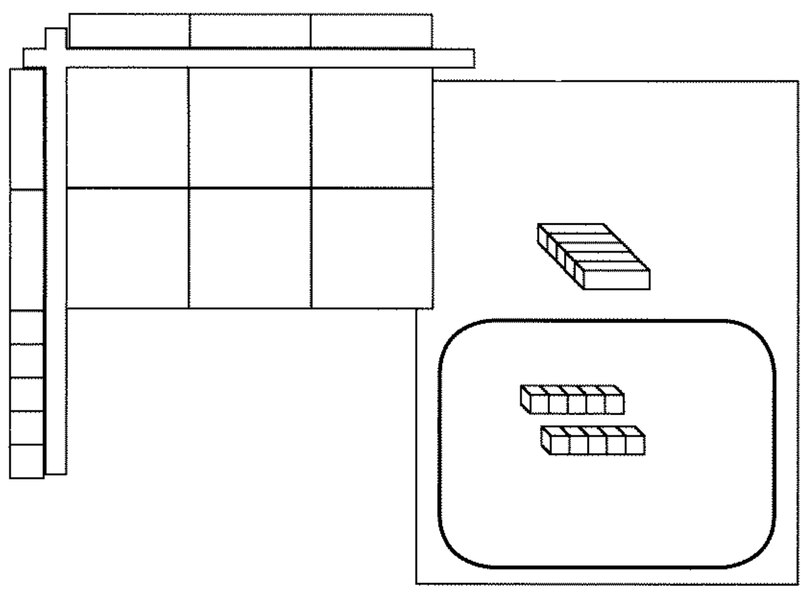
Study this example. Follow along with your blocks.

- As usual, the denominator is placed on the left side of the corner piece. But this time the numerator is placed in the remainder area.

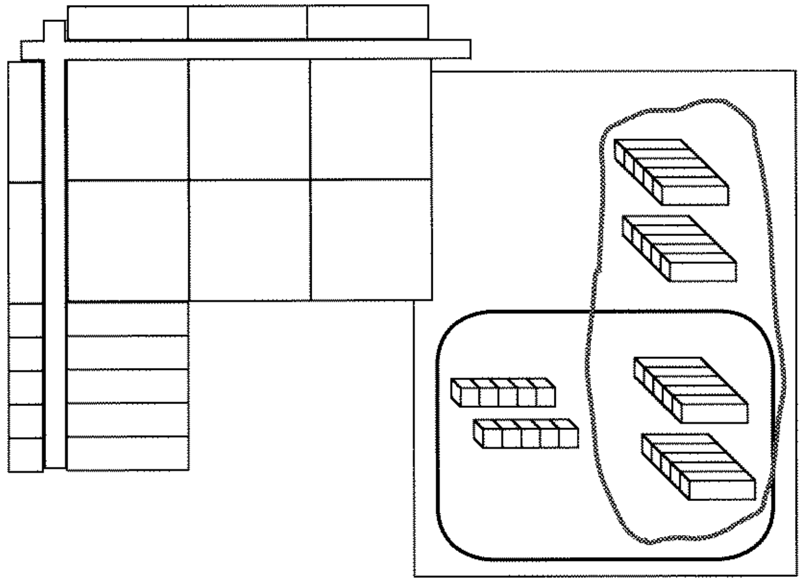
1. Write the division.



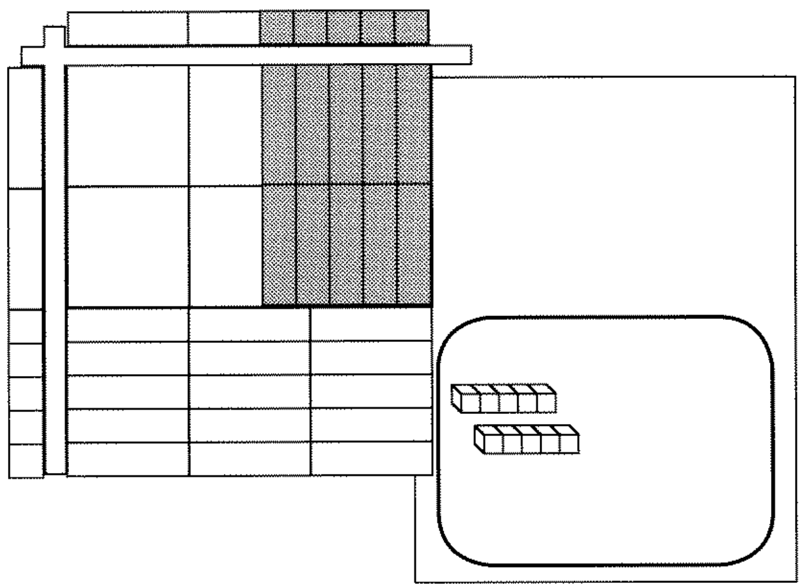
- The first step is to move the x^2-blocks from the remainder area to the inside of the corner piece. Line them up to match the x-blocks on the left. This shows that you need three x-blocks across the top.



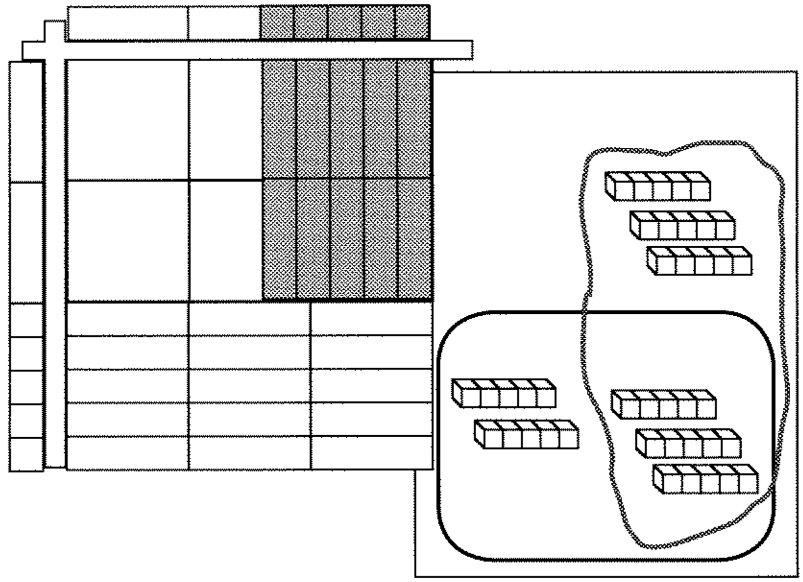
- Now complete the rectangle by moving fifteen *x*-blocks to the inside of the corner piece. Only five are in the remainder area, so zero must be added in the form of 10*x* and –10*x*.



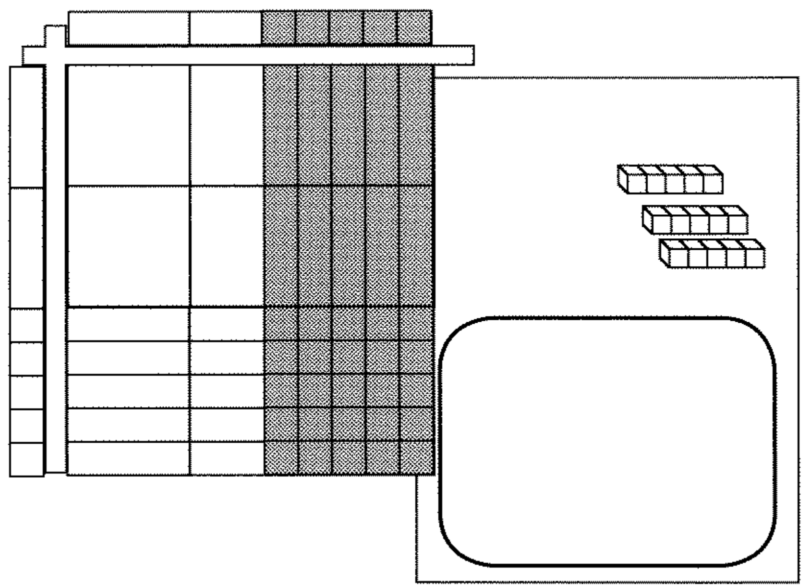
- Be sure to move the *x*-blocks from the minus area to the upstairs of the inside of the corner piece. The other crucial thing to keep in mind is that they must appear vertically, so that they reveal the rest of the quotient. In this case, –5 must appear across the top.



- Again, complete the rectangle. –25 will do it, but only –10 is available in the remainder area. Therefore it is necessary to add zero in the form of 15 and –15.



2. Write the quotient and the remainder.



Use this method to do these divisions.

3. $\dfrac{2x^2 + 5x + 3}{x + 1}$

4. $\dfrac{4x^2 - 8x + 5}{2x - 4}$

5. $\dfrac{y^2 + 10y - 6}{y + 4}$

6. $\dfrac{6x^2 - 10x + 13}{3x - 5}$

☑ Self-check

4. $2x$, r 5

Long Division Without the Lab Gear

- To divide polynomials, you set up the problem like you would set up a **long division** with numbers.

$$2x+5\overline{\smash{)}6x^2+5x-10}$$

- First, divide. To figure out the quotient, start with the question, "How many times $2x$ to get $6x^2$?" and write the answer ($3x$) above the line.

$$\begin{array}{r}3x\\ 2x+5\overline{\smash{)}6x^2+5x-10}\end{array}$$

- Then multiply $3x$ by $(2x + 5)$, and write the answer lined up correctly beneath the numerator. This quantity represents a complete rectangle.

$$\begin{array}{r}3x\\ 2x+5\overline{\smash{)}6x^2+5x-10}\\ -(6x^2+15x)\end{array}$$

- To move it *out of the remainder*, subtract it from the numerator.

$$\begin{array}{r}3x\\ 2x+5\overline{\smash{)}6x^2+5x-10}\\ \underline{-(6x^2+15x)}\\ -10x-10\end{array}$$

- Now repeat the whole process. Divide $2x$ into $-10x$ to find the quotient; multiply to get a complete rectangle; subtract to move it out of the remainder.

$$\begin{array}{rl}3x-5 & \leftarrow \text{divide}\\ 2x+5\overline{\smash{)}6x^2+5x-10} & \\ \underline{-(6x^2+15x)} & \\ -10x-10 & \\ \underline{-(-10x-25)} & \leftarrow \text{multiply}\\ 15 & \leftarrow \text{subtract}\end{array}$$

As expected, this method gives the same answer as the block method. The quotient is $3x - 5$, and the remainder is 15.

Solve these problems. Do each one first with the Lab Gear, then with the method shown above. Compare the way the blocks look to the written way of solving the problem.

1. $\dfrac{2y^2+6y+5}{y+3}$
2. $\dfrac{4x^2-7x+8}{x-1}$
3. Do problems 3-6 from lesson 5 again without blocks. Check to see if you get the same answer.

The following problems cannot be done with the Lab Gear. Use the above method.

4. $\dfrac{x^3+2x^2-5x+9}{x+4}$
5. $\dfrac{4x^3+8x^2-5x-9}{2x^2+x-4}$

☑ Self-check

2. $4x - 3$, r 5

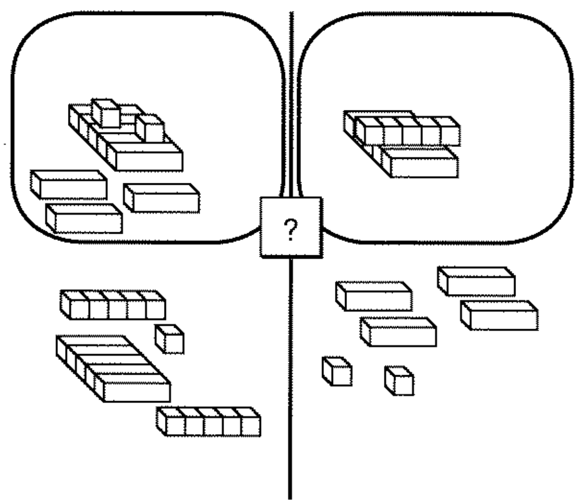
Exploration 4 Which is Greater?

For each of these problems, compare the two expressions by trying various values for x. Good values to try are –2, –1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, and 2. Decide, if possible, which symbol (<, ≤, =, ≥, >) to substitute for the question mark. Explain your answers.

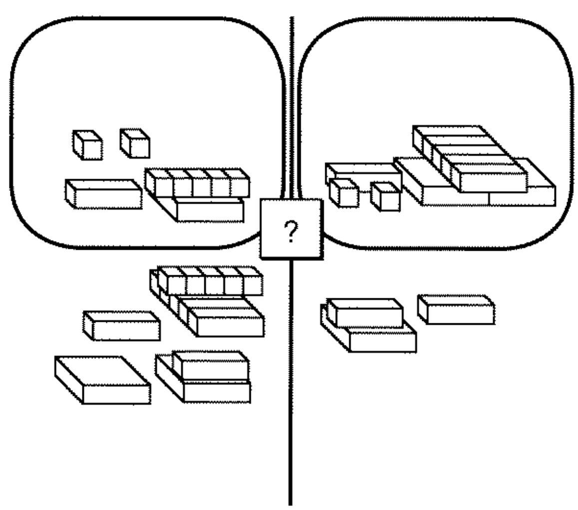
1. x^2 **?** x
2. x^2 **?** $2x^2$
3. x^3 **?** x^2
4. x^3 **?** x

For these problems, write the *Which is Greater?* question that is illustrated by the Lab Gear. Then simplify both sides to answer the question. If it is impossible to tell which is greater, try to say *which values of x make the two sides equal. Which values of x make the left side greater*?

5.



6.



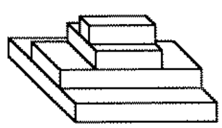
Exploration 5 Perimeter

These problems are about the figure formed by the top faces of the blocks.

1. Arrange three blocks so that the perimeter of the resulting figure is $6x + 2y$. How many solutions can you find?
2. Arrange four blocks so that the perimeter of the resulting figure is $8x + 18$. How many solutions can you find?
3. Arrange five blocks so that the perimeter of the resulting figure is $2y + 2x + 12$. How many solutions can you find?

Exploration 6 Surface Area

Find the surface area of these buildings.

1. 

2.