Chapter 8 Factoring

This chapter motivates the development of factoring techniques by showing students how they can help simplify algebraic fractions and solve some quadratic equations.

New Words and Concepts

The **Zero Product Principle** provides an approach to the solving of quadratic equations. The cancelling of **common factors** is the way to simplify fractions. Both require **factoring**.

While factoring is not an important skill for a physicist or engineer, it does offer useful learning opportunities in an algebra course. First, it constitutes review of the critically important distributive law. Second, because there is no automatic algorithm for factoring, the student learns to use trial and error, and to develop a feel for algebraic manipulation in a puzzle-solving context. Finally, skill at factoring is quite useful in further schoolwork and test-taking.

Teaching Tips

The "Make a Rectangle" Explorations in previous chapters laid the foundation for these lessons. However, they generally limited themselves to binomials with only plus signs. What is new in this chapter, aside from the application of factoring to other problems, is the extension to more complicated cases that involve one or two minus signs. These are quite challenging to work out with the Lab Gear, and some students prefer to do them just by algebra. On the other hand, many students find that there is something exciting about the way the blocks fit perfectly, and the rectangle model holds up even with minus signs.

Lesson Notes

- **Lesson 1,** The Zero Product Principle, page 98: Be sure to precede this lesson with Exploration 1. Note that the equations are given in factored form.
- **Lesson 2,** Reducing Fractions, page 100: Solid understanding of this material, is of course, only possible if one already understands fractions in arithmetic. However, the concrete image of the rectangles with a matching side can go a long way towards clarifying the concept of a common factor, and students who had trouble with simplifying nonalgebraic fractions may be helped significantly by this approach.
- **Lesson 3,** Common Factors, page 102: This lesson is intended to start the work on factoring by defining the terms and the general strategy, while at the same time posing some difficult questions.
- **Lesson 4,** Recognizing Identities, page 104: This is based on the work done in Chapter 6, Lesson 4, as well as various explorations.
- **Lesson 5,** More Factoring, page 106: Straightforward "Make a Rectangle" problems.
- **Lesson 6,** More Difficult Factoring, page 107: Minuses in both factors.
- **Lesson 7,** Even More Difficult, page 108: A minus sign in one of the two factors.
- **Lesson 8,** Factoring Practice, page 109, and **Lesson 9,** Reducing Fractions, page 110: In these lessons, students need to use all the factoring techniques they have learned. Chapter 9, Lesson 2 offers more opportunities for general review of factoring, in the context of solving equations.

Exploration 1 True or False?

Suppose a and b are numbers. Indicate whether these statements are true or false by writing T or F. Explain each answer.

1. If $a = 0$ and $b = 0$, then $ab = 0$
2. If $a \neq 0$ and $b = 0$, then $ab = 0$
3. If $a = 0$ and $b \neq 0$, then $ab = 0$
4. If $a \neq 0$ and $b \neq 0$, then $ab = 0$
5. Change one symbol in statement 4 to make it true. (Do not change it to any of the statements 1-3.)
6. If you know that $ab = 0$ what can you conclude about a and b?

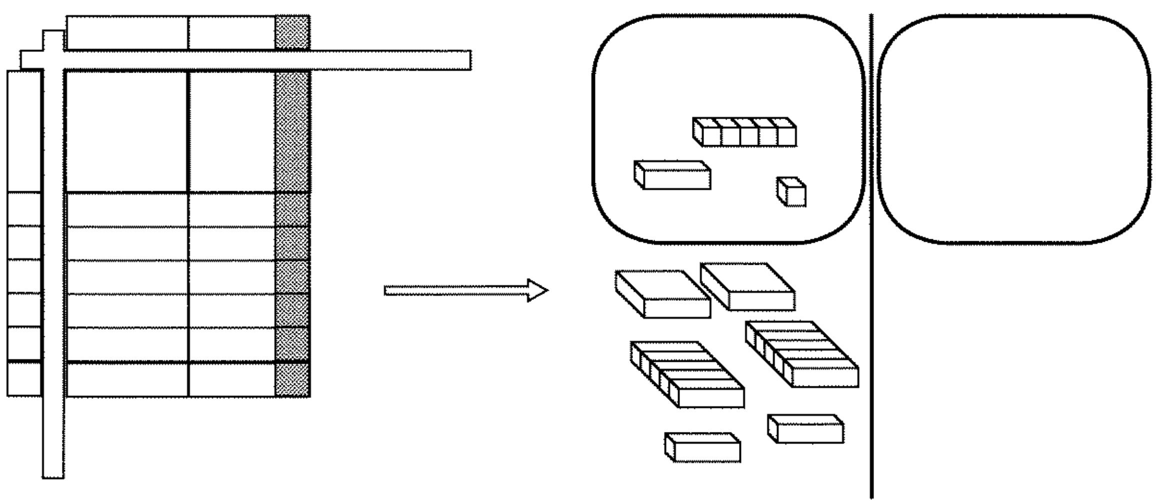
Lesson 1

The Zero Product Principle

Consider using blocks to solve the equation $(x + 6)(2x - 1) = 0$

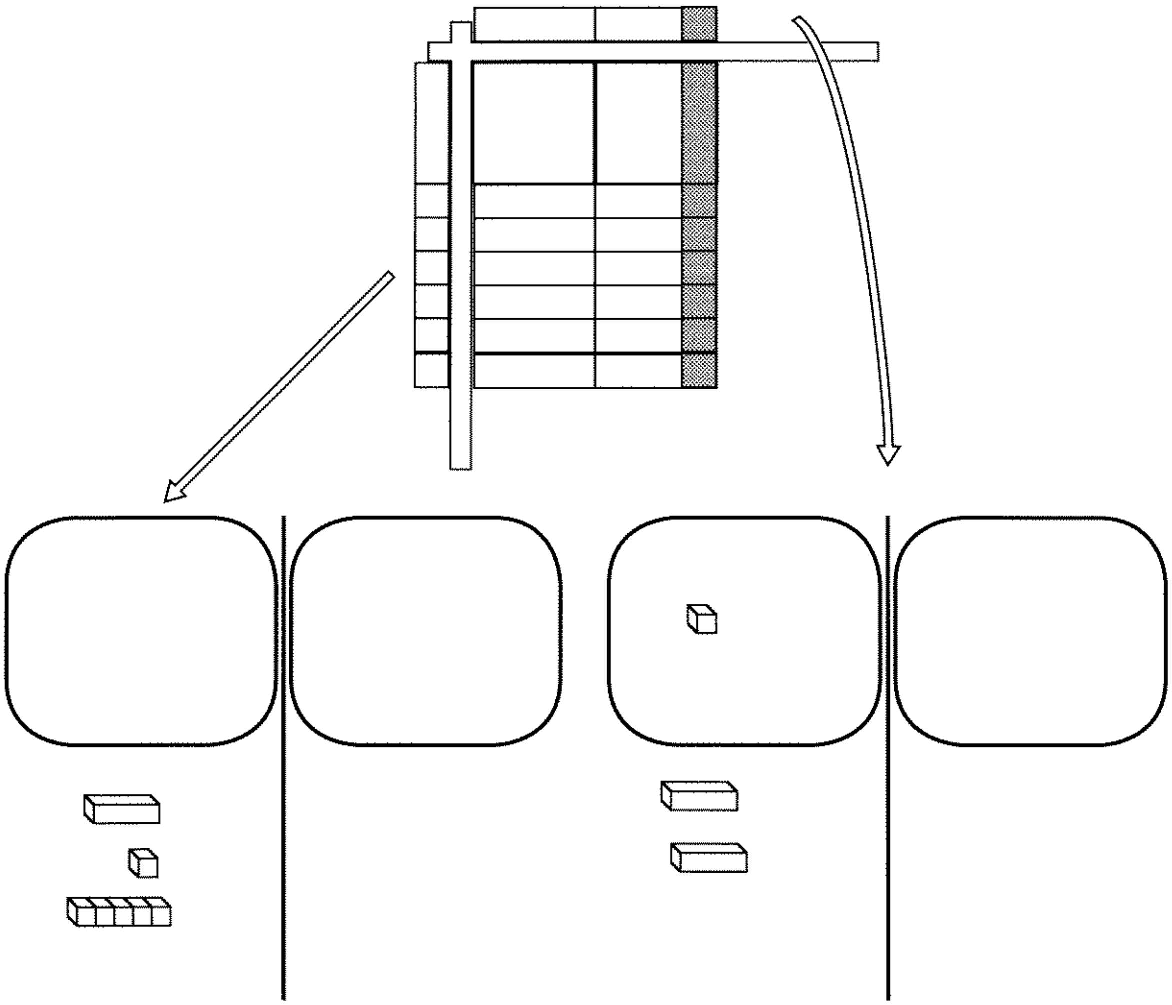
Setting this up with the Lab Gear gives this figure.

1. Explain the figure.



If you try to solve this equation with the methods you learned in chapter 7, you will find that it is not possible. For an equation of this type, we need a whole new approach—one based on the **Zero Product Principle**, which says: *When the product of two quantities is zero, one or the other quantity must be zero.*

Remember, the equation we are trying to solve is $(x + 6)(2x - 1) = 0$. Since the product is zero, you can write these two equations.



$x + 6 = 0$ or, $2x - 1 = 0$

2. You know how to solve these equations. Write the solutions.
3. There are two solutions to the equation $(x + 6)(2x - 1) = 0$. What are they?

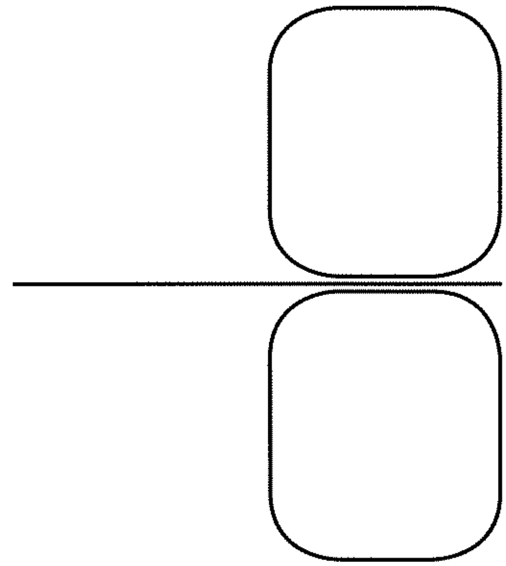
Solve these equations.

4. $(3x + 1)(x + 5) = 0$
5. $(2x + 3)(5 - x) = 0$
6. $(2x - 2)(3x - 1) = 0$

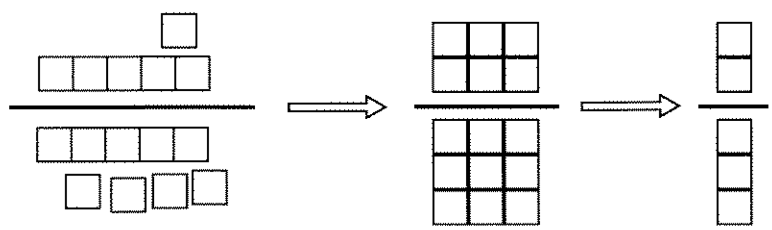
Reducing Fractions

As you know, to reduce fractions, you divide the numerator and denominator by a common factor. For example, $\frac{6}{9}$ can be reduced by dividing 6 and 9 by 3, which gives $\frac{2}{3}$.

This can be modeled with the Lab Gear. To do that, use the workmat turned on its side. Instead of representing an equals sign, the middle line now represents the fraction bar.



Arrange the blocks in the numerator into a rectangle (or square). Do the same in the denominator. If the two rectangles share a side, they can be lined up on either side of the line. Dividing *top and bottom* by the common side, will give you the new numerator and denominator.



Reduce these fractions by making rectangles with your Lab Gear.

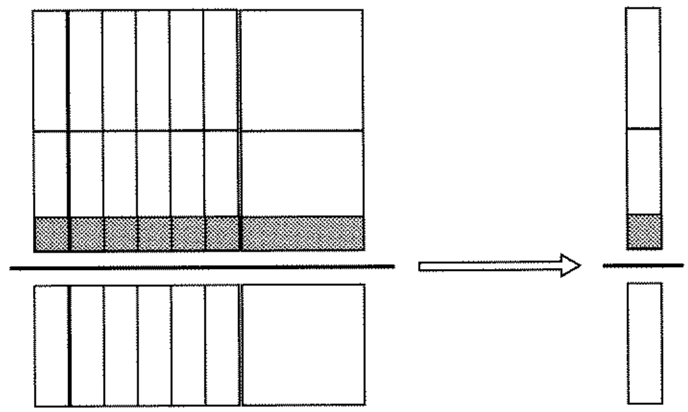
1. $\frac{4}{12}$

2. $\frac{8}{24}$

3. $\frac{12}{18}$

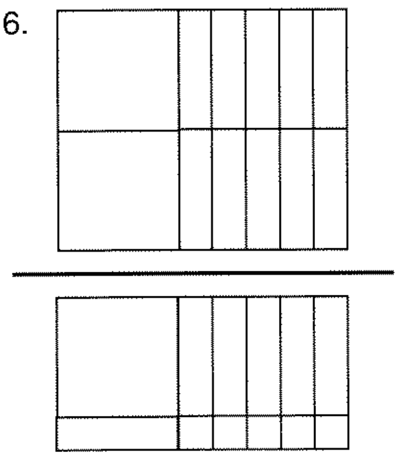
Using the same method, you can reduce algebraic fractions.

4. Name the fraction that is being reduced in this figure.
5. Name the reduced fraction.



Write and reduce these fractions.

6.



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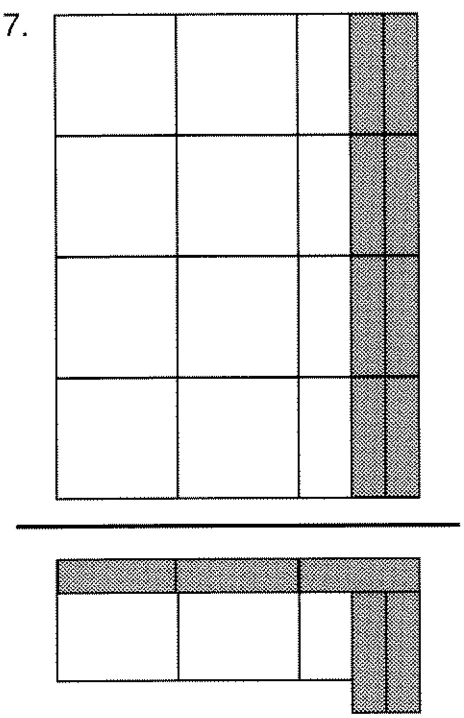

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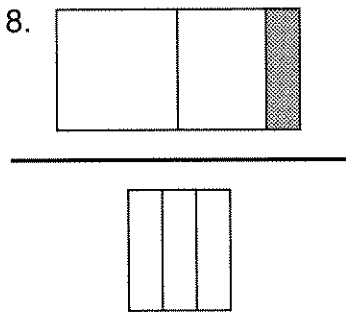


Common Factors

So far in this chapter, you have learned how to solve equations using the Zero Product Principle and you have also learned how to reduce algebraic fractions.

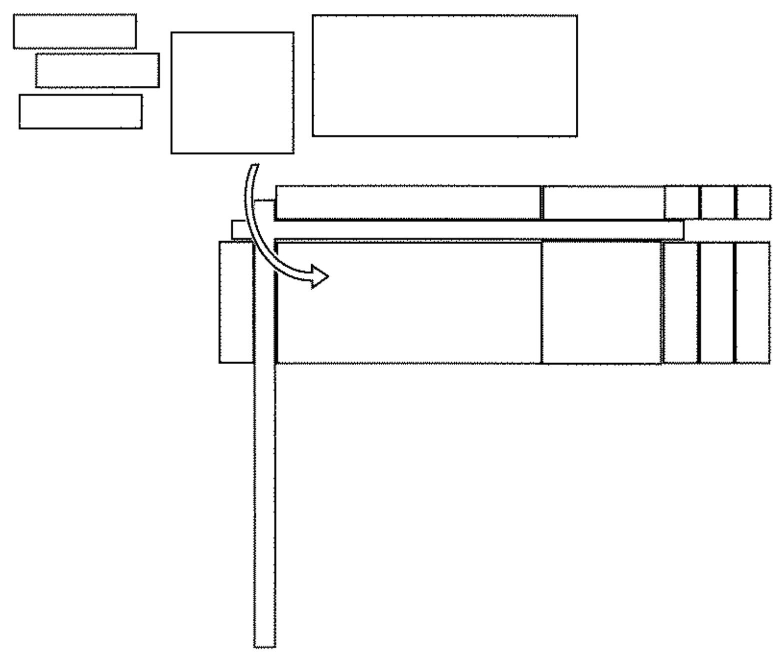
In both cases, using the Lab Gear, the blocks must be arranged in rectangles (or squares). To reduce fractions, you must have a rectangle above the fraction bar, and a rectangle below. To solve equations using the Zero Product Principle, you need a rectangle that is equal to zero.

Unfortunately, blocks are not usually arranged in a rectangle. (In fact, it is often impossible to arrange them that way.) In the rest of this chapter, you will learn some more ways to rearrange blocks into rectangles.

Arranging Lab Gear blocks into rectangles is the same as rewriting polynomials as products. In algebra this process is called **factoring**.

A *factor* of an algebraic expression is an expression that divides it evenly. For example, x is a factor of x^2, since $x \cdot x = x^2$.

1. Write the polynomial (inside the corner piece) that is being factored.
2. Write the two factors of the polynomial.



If you multiply the two factors, using the distributive rule, you would get the original polynomial back. Factoring is applying the distributive rule in reverse.

Use the Lab Gear to factor these polynomials. Not all are possible.

3. $2x^2 - x$
4. $2x^2 + 6x + 1$
5. $x^2 + 2x + xy$
6. $3x^2 - 3x$

In problem 5, x is a factor of x^2, of $2x$ and of xy because it divides each term evenly. We say that x is a *common factor* of the three terms.

The following problems are more challenging. Even though there is no common factor, the blocks can be rearranged into a rectangle (or square). Remember that it is the *uncovered* part of the blocks that needs to make a rectangle.

7. $x^2 + 6x + 9$
8. $y^2 - 2xy + x^2$
9. $y^2 - 4$

If you have trouble, get help from your classmates. Don't worry, you will learn more about how to factor these polynomials in the next lesson.

Exploration 2 Inequalities

For each equation, find the values of x that make it true.

1. $x + 5 > 1$
2. $x - 5 > 1$
3. $x + 6 > 0$
4. $x - 6 > 0$
5. $x - 1 > 5$
6. $x + 1 > 5$
7. $x - 5 > -1$
8. $x + 5 > -1$
9. $-x > 6$

10. Check your answer to problem 9 by carefully trying various values for x. (Most algebra students get this problem wrong, and come up with the answer $x > -6$.)

11. For what values of x is $-x > -6$?

12. Explain how problems 9 and 11 differ from problems 1-8. Tell how to handle this situation when trying to solve an inequality.

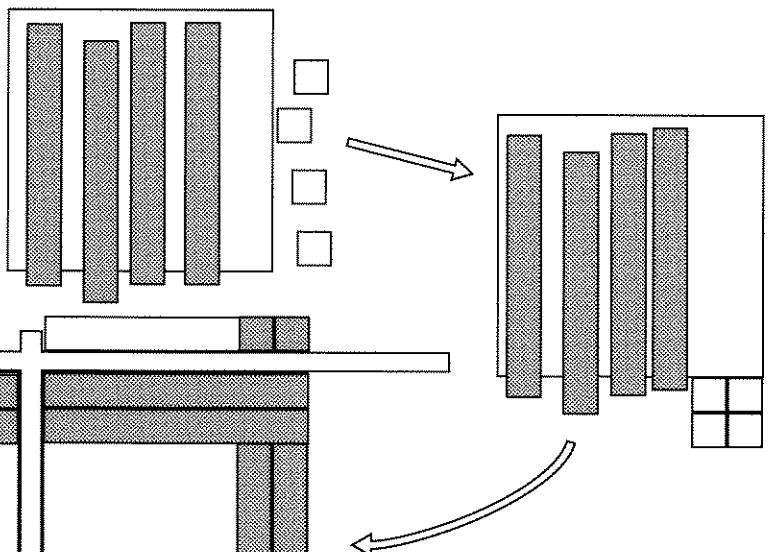
Recognizing Identities

When there is no common factor, it is still sometimes possible to rearrange the blocks into a rectangle (or square). In these problems, factor the polynomials by arranging the blocks into a *square*. Not all are possible.

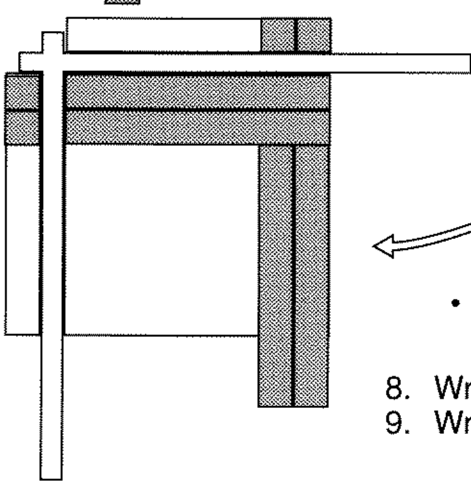
1. $x^2 + 4x + 4$
2. $9x^2 + 6x + 1$
3. $x^2 + 4x + 9$
4. $x^2 + 6x + 3$
5. $x^2 + 2xy + y^2$
6. $2x^2 + 8x + 16$
7. Explain what makes some of these problems possible, and others not.

These problems were based on the identity $(a + b)^2 = a^2 + 2ab + b^2$, which you used in Chapter 6. It is a little more difficult when there are minuses.

Look at this example.



• This figure shows how to set up the downstairs blocks first.



• Next, rearrange the upstairs blocks. Notice that the uncovered area is a square.

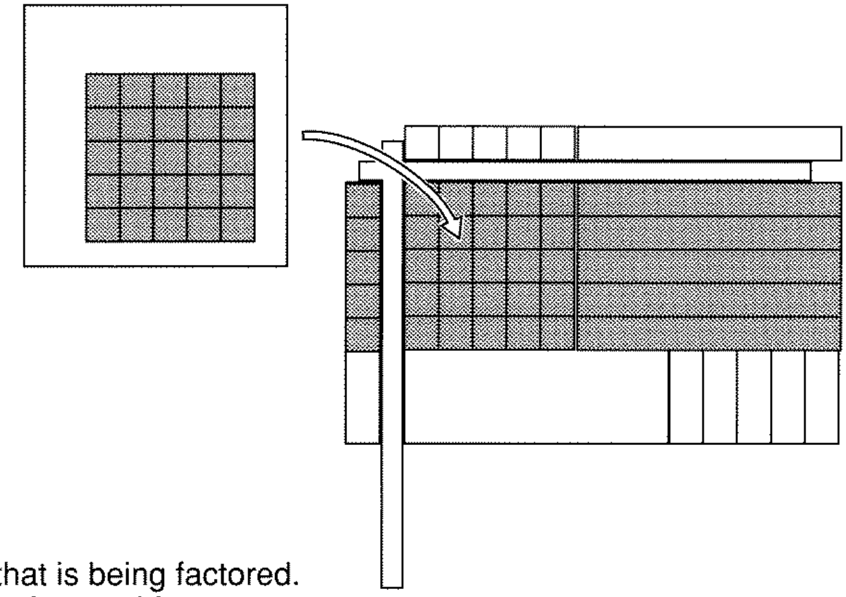
8. Write the polynomial that is being factored.
9. Write the two factors of the polynomial.

Use the same method to factor these polynomials. Not all are possible.

10. $4y^2 - 4y + 1$
11. $y^2 - 3y + 9$
12. $y^2 - 6y + 9$
13. $y^2 - 10y + 25$
14. Each of the polynomials in problems 1, 2, and 5 is the *square of a sum*. How would you describe each of the polynomials in problems 10, 12, and 13? Write this identity in terms of variables *a* and *b*.

This example shows a type of factoring based on the third important identity—the *difference of two squares*.

In this case, the only way to get a rectangle is to *add zero*, with $5y$ and $-5y$, and to arrange the blocks as shown. The uncovered rectangle has dimensions $y + 5$ by $y - 5$.



15. Write the polynomial that is being factored.
16. Write the polynomial in factored form.

Use this method to factor these polynomials.

17. $x^2 - 1$
18. $y^2 - 4$
19. $y^2 - x^2$
20. Go back to problems 7-9 in Lesson 3, page 103. If you had trouble factoring these expressions, try them again.

Factor these polynomials. Use any of the methods you have learned. Not all are possible.

21. $4x^2 + 4x + 1$
22. $x^2 + 2x$
23. $y^2 - 9$
24. $y^2 - 8y + 16$
25. $x^2 + 9$

Solving Equations

Use any method to solve these equations. Remember the Zero Product Principle.

26. $y^2 - 16 = 0$
27. $5x + 2 = x - 6$
28. $2x^2 + 6x = 0$
29. $4x^2 + 12x + 9 = 0$
30. $y^2 - 8y + 16 = 0$

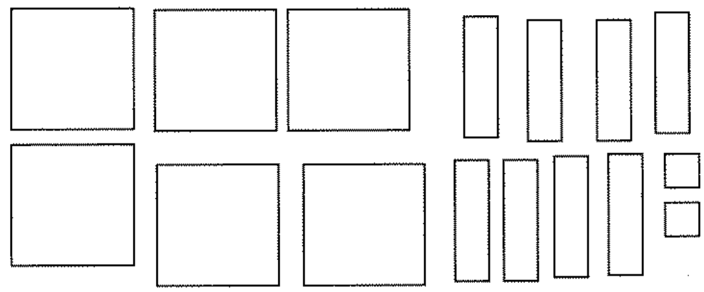
☑ Self-check

21. $(2x + 1)^2$

More Factoring

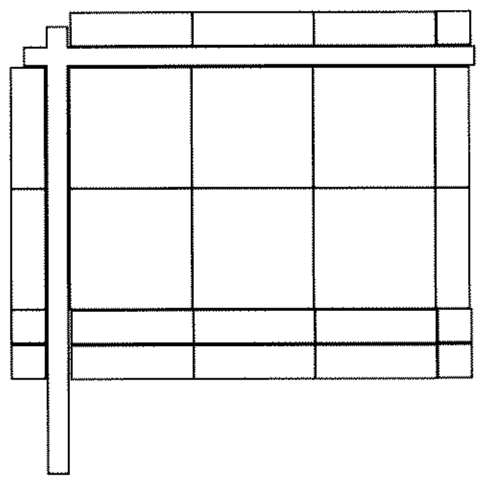
You have learned to factor polynomials that have a common factor, or that are examples of a standard identity. Some other polynomials can be factored by using *trial and error*.

Look at this example. Show it with your blocks. Try to find the factors without looking at the solution below. Use trial and error.



Check your work with this figure.

1. Write the polynomial that is being factored.
2. Write the polynomial in factored form.



Using trial and error, factor these polynomials. One is impossible.

3. $9x^2 + 6x + 1$
4. $2x^2 + 5x + 2$
5. $x^2 + 7x + 10$
6. $3x^2 + 4x + 2$
7. $2x^2 + 11x + 5$

☑ Self-check

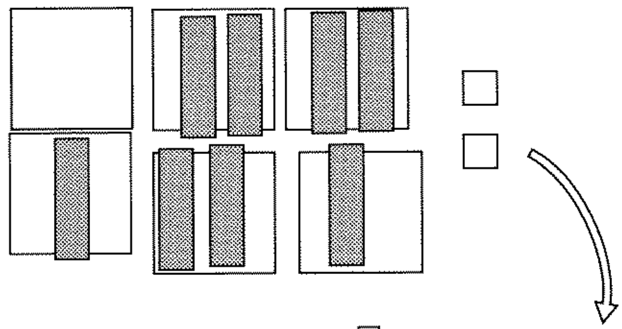
4. $(2x + 1)(x + 2)$

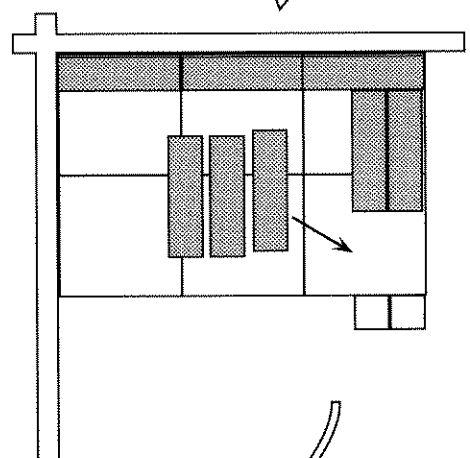
More Difficult Factoring

As you might guess, minus signs make it more difficult to factor polynomials.

Look at this figure. Show it with your blocks.

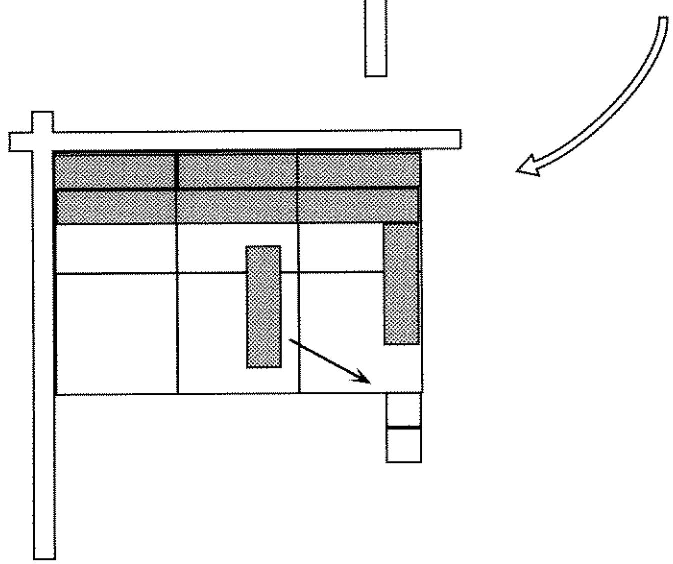
1. Write the polynomial that is being factored.





- First arrange the downstairs blocks. Remember that the goal is to get the uncovered part to be a rectangle.

The first try shown in this figure does not work, because even after sliding two more *x*-blocks to the bottom right, there would be one left over, ruining the rectangle.



- However, by rearranging the yellow blocks and the upstairs *x*-blocks, it is possible to solve the problem.

2. Write the polynomial in factored form.

Using this method, factor these polynomials.

3. $y^2 - 6y + 5$
4. $y^2 - 2y + 2x - xy$
5. $4x^2 - 12x + 5$
6. $3x^2 - 5x + 2$

☑ Self-check

3. $(y - 1)(y - 5)$

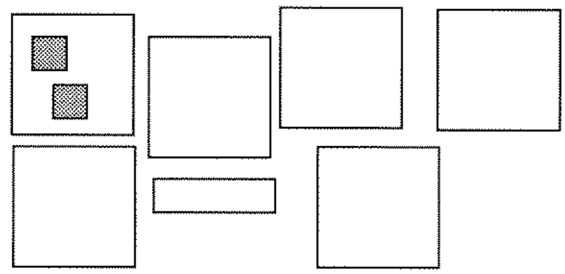
Even More Difficult

1. Multiply $(2x - 2)(3x + 1)$ with the Lab Gear. Notice that at the end, you can cancel matching upstairs and downstairs blocks ($2x$ and $-2x$). Write the answer. Now, reverse this canceling to factor the answer.

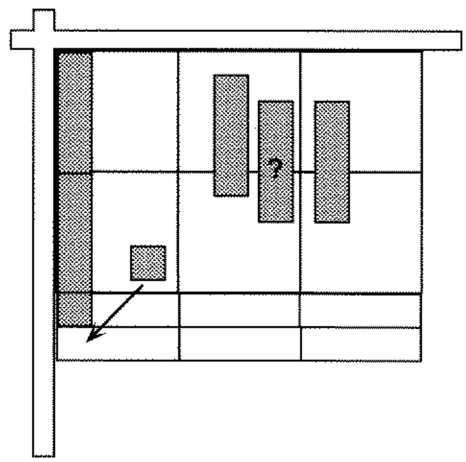
When trying to factor polynomials with the Lab Gear, it is sometimes necessary to "uncancel" upstairs and downstairs blocks, or once again, to *add zero*.

Look at this example. Show it with your blocks.

2. Write the polynomial that is being factored.

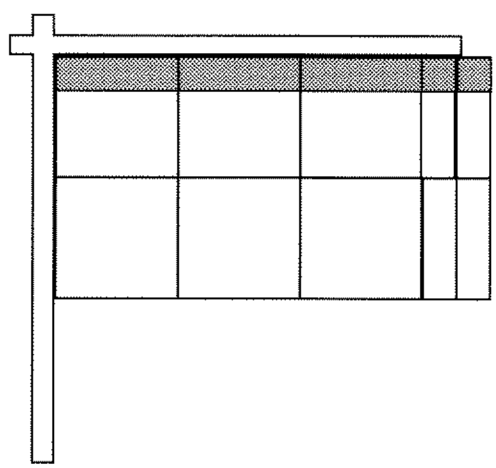


The first try at factoring, involving an additional $5x$ and $-5x$ does not work, since we have three upstairs x-blocks left which prevent us from getting an uncovered rectangle.



To find the solution we must add $3x$ and $-3x$.

3. Write the factors of the polynomial.



For these expressions, multiply, cancel, and write the answer. Then *uncancel* and factor. In other words, recreate the rectangle.

4. $(y - 3)(y + 1)$
5. $(2x - 2)(x + 5)$

Using this method, factor these polynomials.

6. $3x^2 + x - 10$
7. $2x^2 + 2x - 12$
8. $4x^2 - 4x - 3$
9. $y^2 - xy - 5x + 5y$

Factoring Practice

Factor these polynomials. One is impossible.

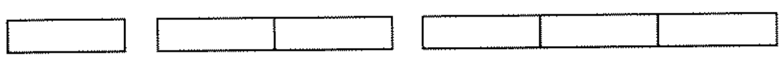
1. $xy + 6y + y^2$
2. $y^2 - 16$
3. $3x^2 + 13x - 10$
4. $4x^2 + 8x + 4$
5. $2x^2 + 2x + 1$
6. $y^2 - 5y + 6$
7. $y^2 - 4y + 4$
8. $x^2 + 8x + 12$

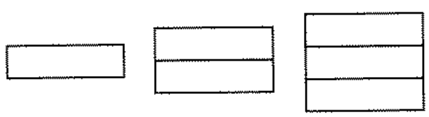
☑ Self-check

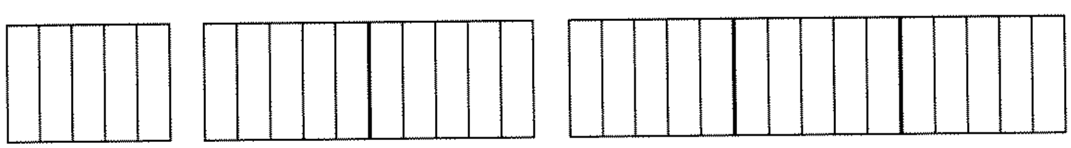
3. $(3x - 2)(x + 5)$

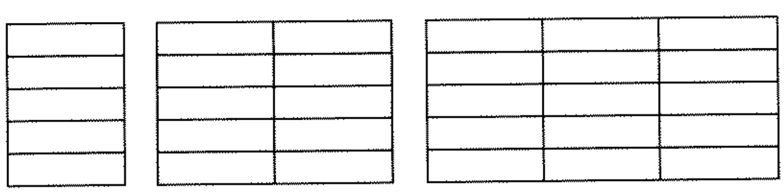
Exploration 3 Perimeter

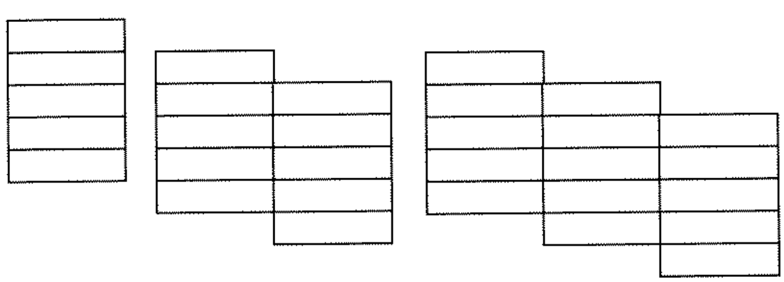
Look at each sequence. Think about how it continues, following the pattern. Write the perimeters of the figures given, then the perimeter of the fourth one, the tenth one, and the hundredth one.

1. 

2. 

3. 

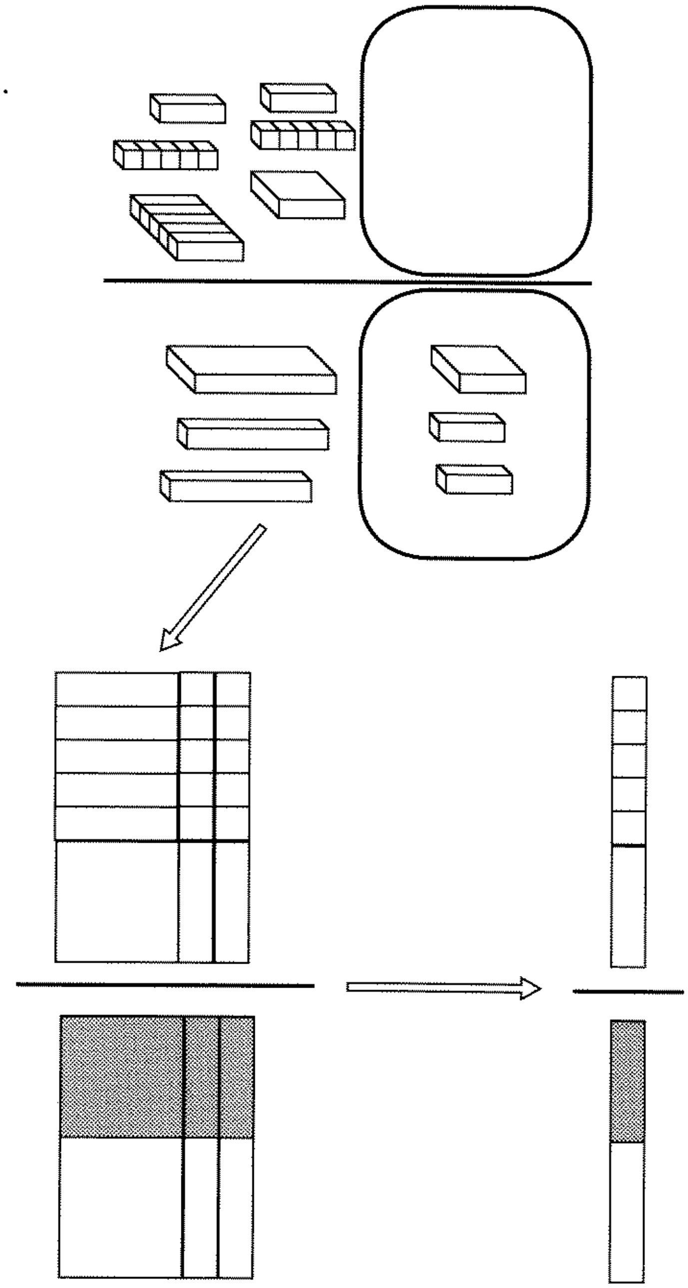
4. 

5. 

Reducing Fractions

To reduce fractions, remember that you can use the Lab Gear to make rectangles that match in one dimension.

1. Write the fraction that is being reduced.
2. Write the fraction in its factored form.
3. Write the reduced fraction.



Use this method to reduce these fractions. One is impossible.

4. $\dfrac{5y-5x}{y^2-x^2}$

5. $\dfrac{x^2+3x+2}{x^2-4}$

6. $\dfrac{x^2+5x}{2x^2+xy}$

7. $\dfrac{y^2-7y+12}{y^2-6y+9}$

8. $\dfrac{x+5}{2x^2-5x-25}$

☑ Self-check

7. $\dfrac{y-4}{y-3}$

Exploration 4 Surface Area

Look at each sequence. Think about how it continues, following the pattern. Write the surface areas of the figures given, then the surface area of the fourth one, the tenth one, and the hundredth one.

1. 

2. 

3. 

4.