Urban School / Math 2B

Introduction to Proof

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Proving Something is False

Conjectures can be proved **false** with a single *counter-example*.

For example:

In a quadrilateral, if all angles are congruent, then all sides are congruent.

This is false, because a rectangle that is not a square constitutes a counter-example: all its angles are congruent, and yet all the sides are not congruent.

Proving Something is True

Conjectures can be proved **true** by using a logical argument, based on known facts. When a conjecture has been proved true, it is called a *theorem*.

A proof is a logical argument. In math, something is considered true if it has been proved. It is not enough for something to *seem* true. In writing a proof, you can only use facts that have previously been proved, or facts that are assumed true without proof. In this class, we will assume the following facts are true without proving them:

- Vertical angles are congruent.
- If a line crosses two parallel lines, then corresponding angles are congruent. (also alternate interior)
- If the corresponding (or alternate interior) angles are congruent, then the lines are parallel.
- The sum of the angles in a triangle is 180°.
- SSS, SAS, ASA for congruent triangles.
- SSS, SAS, AA for similar triangles.

Ingredients of a proof

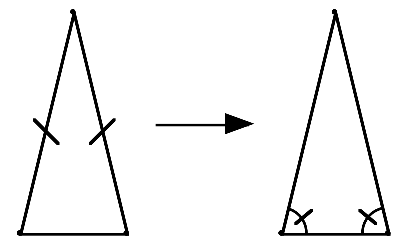
Setup:

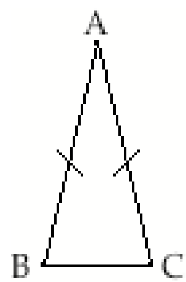
- A statement of what you are trying to prove,
- An "if, then" diagram,
- A figure, and
- A "given, prove" statement, based on the figure

Example

The Isosceles Triangle Theorem: in a triangle, if two sides are congruent, then the angles opposite them are congruent.

If-then diagram:





Given: In ΔABC, $AB = AC$
Prove: $\angle ABC \cong \angle ACB$

1. You will write a proof based on drawing an auxiliary (i.e. helping) line.
   a. On the figure, mark the midpoint of segment BC. Label it M.
   b. Mark the resulting equal segments.
   c. Join segment AM with a line segment. (AM is called the *median*.)
   d. Write a proof that the angles are congruent, giving reasons for all your statements. (Hint: use congruent triangles.)

2. Another way to state this theorem is "The base angles of an ________________ are ________". Fill in the blanks.

Proving the Converse

The *converse* of a theorem is the statement you get when you reverse the "if" and "then" parts. It needs its own proof, because it is not always true.

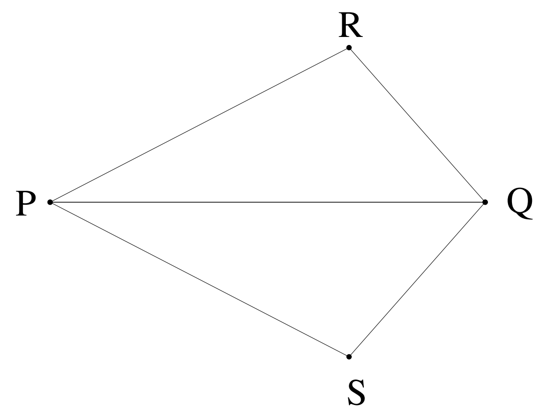
Converse of the Isosceles Triangle Theorem: In a triangle, if two angles are congruent, then the sides opposite them are congruent.

3. a. Draw an "if, then" diagram.
   b. Draw a figure.
   c. Write a "Given" and a "Prove" based on the figure.

4. Write a proof of the Converse of the Isosceles Triangle Theorem, using an *altitude*. (An altitude is the perpendicular dropped from a vertex to the opposite side.)

Two Formats for Proof

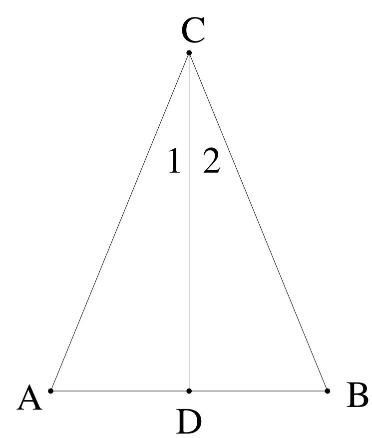
1. This is an example of a proof in *paragraph form*. This is the form mathematicians use. Examine the given information and mark the diagram appropriately. Then fill in the blanks in the proof.
   Given: RQ=SQ and RP=SP.
   Prove: ∠ R = ∠ S.



The given information is: _______________ and __ ____________. PQ=PQ because __________. So ΔPRQ≅ ΔPSQ because of the ____________ property for congruent triangles. Therefore, _______ ______ because corresponding parts of congruent triangles are congruent.

2. This is an example of a proof in *two-column form*. Some teachers and students prefer this form. Examine the given information and mark the diagram appropriately. Then fill in the blanks in the proof.

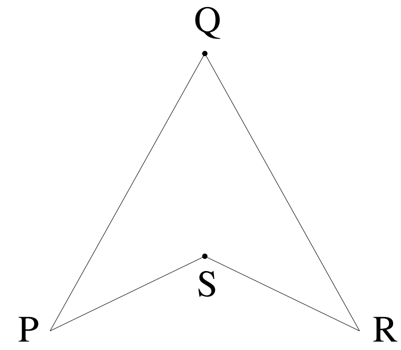
   Given: AC=BC and ∠ 1 = ∠ 2.
   Prove: ∠ A = ∠ B.



Proof:

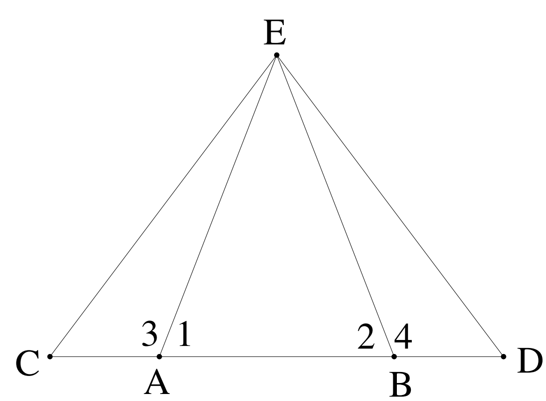
Statements	Reasons
1. AC = BC	1.
2.	2. Given
3. CD = CD	3.
4.	4. by SAS property
5. ∠A = ∠ B	5.

3. Suppose that in the quadrilateral below, we are given that PQ = RQ and PS = RS. Prove that $\angle P = \angle R$. Be sure to mark the diagram and restate the given information and what you are trying to prove.

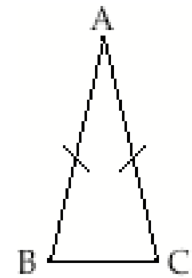


4. Given: AC = BD and $\angle 1 = \angle 2$.

Prove: $\Delta CAE \cong \Delta DBE$.

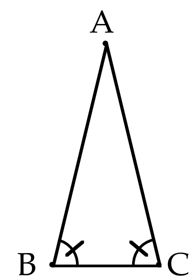


Incorrect Proofs



1. What's wrong with these proofs of the Isosceles Triangle Theorem?
   a. Connect A to a point P on BC. Obviously AP=AP. It is given that AB=AC. When two sides are equal, the third sides must be equal, so BP = PC. We have SSS, so $\Delta APB \cong \Delta APC$, and the corresponding parts are equal. So $\angle ABC = \angle ACB$.
   b. The base angles must be equal, because if they weren't, the triangle would not be isosceles.

2. Write a correct proof of the Isosceles Triangle Theorem without looking at your notes.



3. What's wrong with these proofs of the converse of the isosceles triangle theorem?
   a. Draw the line from A to the midpoint M of BC. Obviously, AM=AM. BM=MC since M is the midpoint. $\angle ABC = \angle ACB$ is given. So $\Delta AMB \cong \Delta AMC$, and the corresponding parts are equal, so AB=AC.
   b. Draw the line from A to the midpoint M of BC. Obviously, AM=AM. BM=MC since M is the midpoint. AB = AC since the triangle is isosceles. So we have SSS, and the corresponding parts are equal, so AB=AC.
   c. Drop a perpendicular from A to BC. Say it meets BC at point H. Obviously, AH=AH, and BH = HC. $\angle AHC = \angle AHB = 90°$. Therefore $\Delta AHB \cong \Delta AHC$ by SAS, and the corresponding parts are equal, so AB=AC.
   d. Drop a perpendicular from A to BC. Say it meets BC at point H. When two angles are equal, so are the third angles. So $\Delta AHB \cong \Delta AHC$ by AAA, and the corresponding parts are equal, so AB=AC.

4. Write a correct proof of the converse of the Isosceles Triangle Theorem without looking at your notes.

Triangle Proofs

A) What are the three ways to show that two triangles are congruent to each other?

B) What do you know about the corresponding parts of two congruent triangles?

C) What is a conjecture? A theorem?

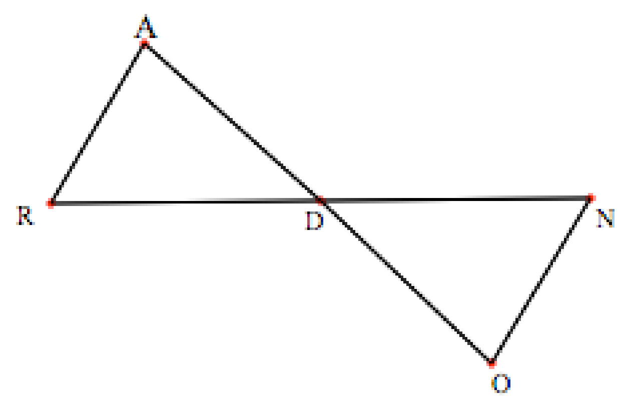
D) State the Isosceles Triangle Theorem:

E) Sketch an if-then diagram to illustrate the Isosceles Triangle Theorem.

For each of the following, write a proof in either two-column or paragraph form, on a separate sheet of paper. Be sure that each of your arguments is supported logically by a reason. Also be sure to watch out for the mistake of making assumptions.

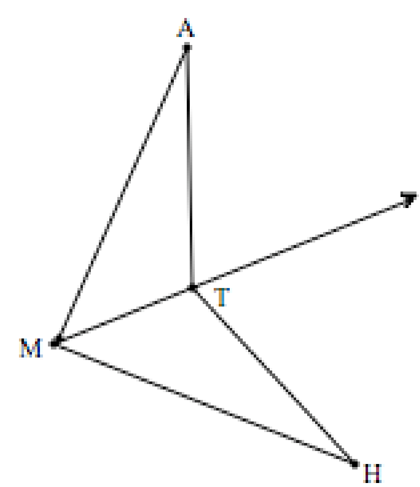
1. Given: ∠ A and ∠ O are right angles and Segment AO bisects Segment RN.

   Prove: ΔRAD ≅ ΔNOD.

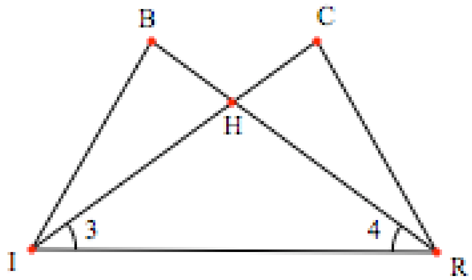


2. Given: Ray MT bisects ∠ M and MA = MH.

   Prove: ΔMAT is congruent to ΔMHT.



3. Given: $\angle 3 = \angle 4$. BH = HC.
   Prove: $\Delta BHI \cong \Delta CHR$.



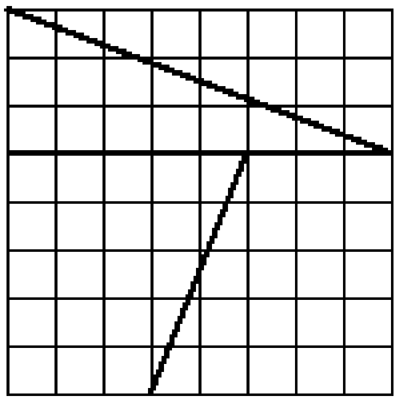
4. Prove: Two right triangles are congruent if the legs of one are equal to the legs of the other (LL).

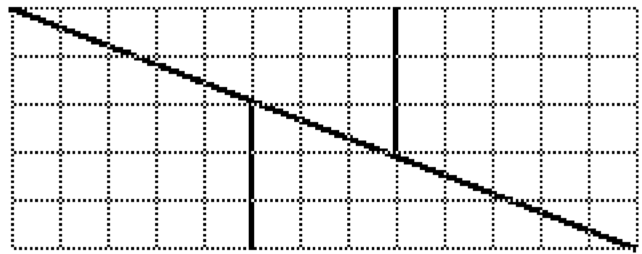
5. Prove: Two right triangles are congruent if they have one equal leg and equal hypotenuses (HL).

6. Prove: Two right triangles are congruent if they have one equal acute angle and equal hypotenuses (AH).

Impossible!

Lewis Carroll liked this puzzle:





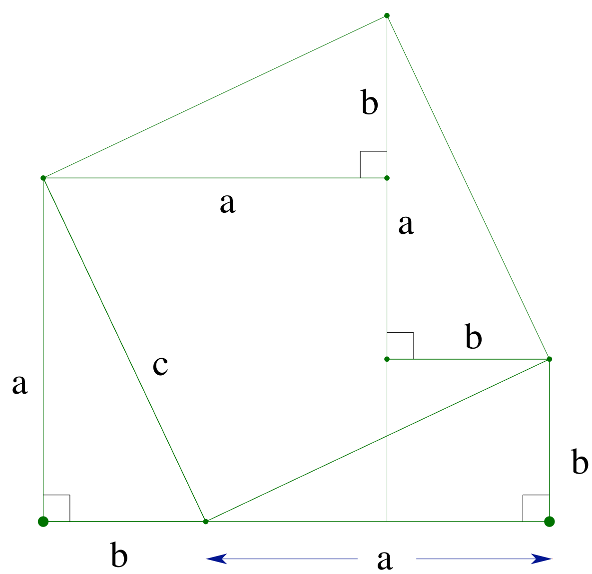
1. What is the area of the square?

2. The pieces of the square can be rearranged to make the figure on the right. What is its area?

3. How do you explain the apparent contradiction?

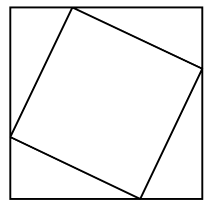
This puzzle shows that just because something looks true, it is not necessarily so! This is why proofs need to be carefully constructed to be completely logically tight. In the example above, we assumed the pieces fit together as shown, without proving it, which led us to an impossible situation.

4. Use this figure to prove the Pythagorean theorem rigorously:



The Tilted Square

In Math 2A, you learned how to create a "tilted square" on your 10x10 geoboard. It looked something like this:

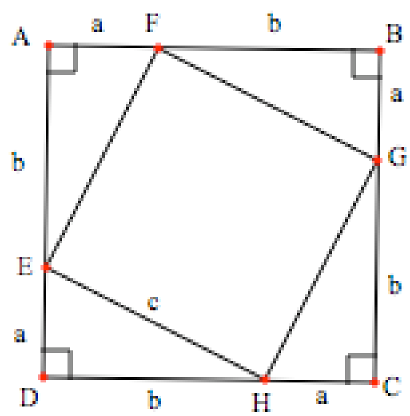


Well, how do you know that the tilted figure is actually a square? That is, how do you know for certain that it has *four right angles* and *four congruent sides*? Appearances can be deceiving!

1. Use your geoboard to create a tilted square that is different from that of your neighbor. Prove that it is indeed a square. Make sure you use the full definition of square and format your proof appropriately.

2. Write down the slopes of each side of your tilted square. Find the relationships that consistently hold for every tilted square at your table. Do the slope relationships hold for a non- tilted square?

3. Write a formal proof for the general case. Given that ABCD is a square with side a+b, prove that EFGH is a square. (Do not use the Pythagorean theorem!)

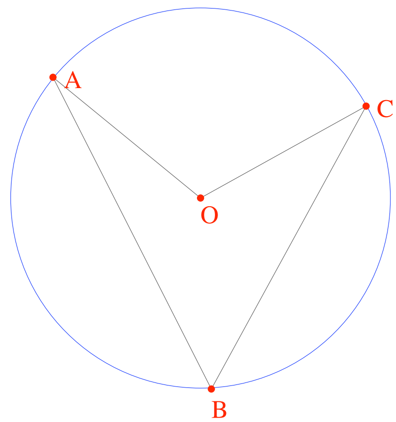


4. Find the slopes of the following segments: EF, FG, GH, and HE. How are they related?

5. a. What is the area of ABCD? What is the area of EFGH?
   b. If you take the area of ABCD and subtract the areas of the four right triangles, what's left over? Use this approach to prove the Pythagorean theorem.

Reviewing Inscribed Angles

In Math 2A, we saw that the measure of an inscribed angle is ___________ the size of its ______________ arc and corresponding _____________ angle. This was called the Inscribed Angle Theorem.



$\angle ABC =$ _____$\angle AOC$

and

$\angle ABC =$ _____$\frown AC$

We will now formally prove this theorem. (Of course, we cannot use the theorem while proving it! We will use basic theorems we know about triangles.)

1. Extend BO on the figure. Call the intersection with the circle D.
2. Explain why $\angle DOC = 2 \cdot \angle DBC$ (Hints: what kind of triangle is ΔBOC? What is $\angle DOC$ for that triangle?)
3. Explain a similar result about $\angle DOA$.
4. Now do some algebra: what is $\angle DOC + \angle DOA$?
5. Use what you figured out in #1-4 to write a proof of the Inscribed Angle Theorem.

6. The proof you wrote would not work if the center of the circle is outside the inscribed angle, as in this figure:
   Bonus: write a proof for this situation. Hint: you still need point D as above, but you have to use subtraction instead of addition.