Algebra and Geometry for All Students!

Henri Picciotto

henri@MathEducation.page

www.MathEducation.page

blog.MathEducation.page

BlueSky, Mathstodon, Twitter: @hpicciotto

Are students getting too much math?

Too much math!

- Where does this charge appear?
  - *Harper's* (twice in the past few years!)
  - Freakonomics Radio (NPR, podcast)
  - Op eds about standards and frameworks
  - Sometimes even within our profession

- Usually focuses on Algebra 2

They may have a point!

Example:

Where does the quadratic formula come from?

if $0 = ax^2 + bx + c$

then complete the square...

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$-\frac{c}{a} = x^2 + \frac{b}{a}x$$

$$-\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

$$-\frac{c}{a} + \left(\frac{b^2}{4a^2}\right) = \left(x + \frac{b}{2a}\right)^2$$

$$-\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \sqrt{\left(x + \frac{b}{2a}\right)^2}$$

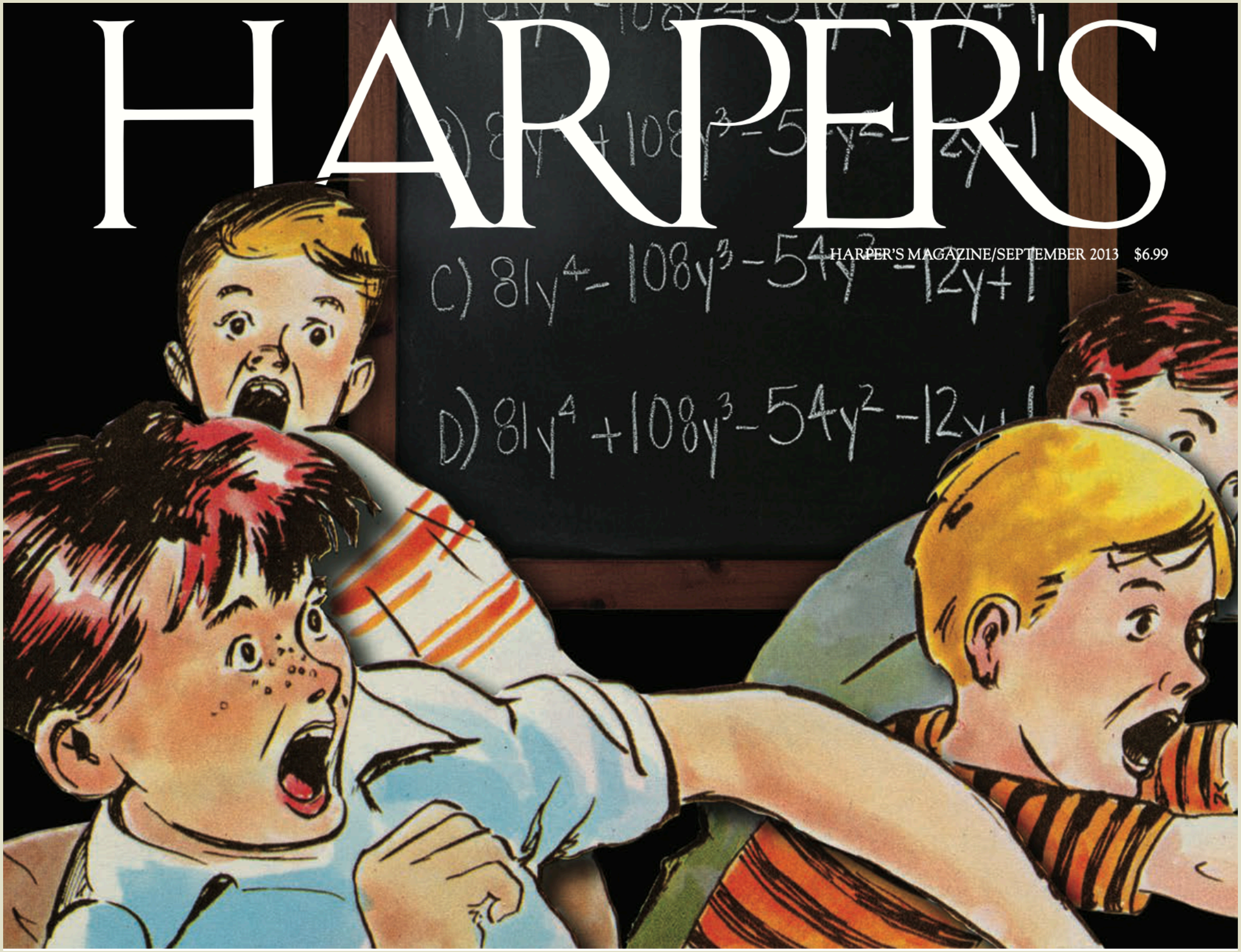
$$\pm\frac{\sqrt{b^2 - 4ac}}{2a} = x + \frac{b}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{So } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad !!$$

HARPER'S

HARPER'S MAGAZINE/SEPTEMBER 2013 $6.99



The problem: symbol manipulation which carries no meaning for most students.

When will I ever use this?

$$\frac{(x-1)(x+y+2)}{x^2-4}$$

Am I supposed to FOIL ?

Still, Algebra 2 is an essential prerequisite to further work in STEM:

- ◊ Physics
- ◊ Chemistry
- ◊ Biology
- ◊ Engineering
- ◊ Math
- ◊ Statistics
- ◊ Computer Science
- ◊ Data Science

Still, Algebra 2 is an essential prerequisite to further work in STEM:

- ◊ Physics
- ◊ Chemistry
- ◊ Biology
- ◊ Engineering

- ◊ Math
- ◊ Statistics
- ◊ Computer Science
- ◊ Data Science

Still, Algebra 2 is an essential prerequisite to further work in STEM:

- ◊ Physics
- ◊ Chemistry
- ◊ Biology
- ◊ Engineering

- ◊ Math
- ◊ Statistics
- ◊ Computer Science
- ◊ Data Science

"The ability to gather and analyze massive amounts of data is indeed transforming our society. But ***'data science' -computer science, statistics, and artificial intelligence- is built on the foundations of algebra, calculus, and logical thinking."***

—Barak, Cohen, Mims, and Nelson (and 1782 other STEM practitioners and educators, 2022)

"The ability to gather and analyze massive amounts of data is indeed transforming our society. But ***'data science' -computer science, statistics, and artificial intelligence- is built on the foundations of algebra, calculus, and logical thinking."***
—Barak, Cohen, Mims, and Nelson (and 1782 other STEM practitioners and educators, 2022)

*"**Data science draws on concepts from mathematics, statistics, computing,*** as well as content knowledge of the domain within which a problem is situated."
—NCTM (2024)

- I support introducing students to data analysis. (I taught a high school data course for decades.)
- *But it should not be at the expense of teaching math!*
- Ideally, it should be done in concert with science and social studies teachers.

If we were to abolish Algebra 2, or make it optional, *it would not disappear from suburban and private schools.*

It would take us back to the days of "STEM is for the white middle class".

Interested in STEM

Interested in STEM

May be interested in STEM someday

Not every student is interested in a STEM career, but some may change their mind in the future.

We should not deny them that opportunity.

What should we do?

Speed and accuracy in computation are no longer legitimate priorities.

What should we do?

Speed and accuracy in computation are no longer legitimate priorities.

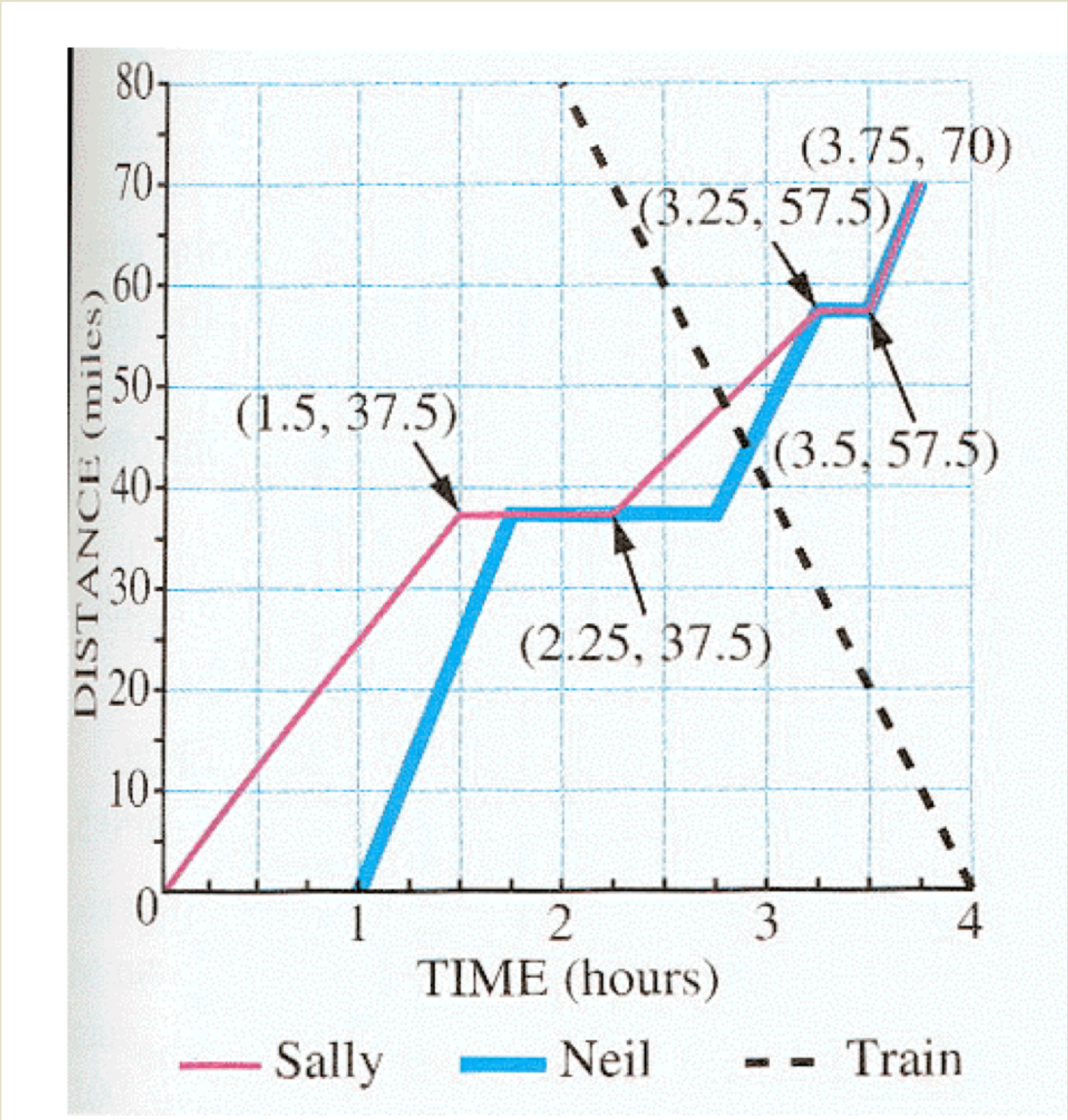
Let's put *meaning* in our Algebra courses!

Build abstraction on a concrete foundation

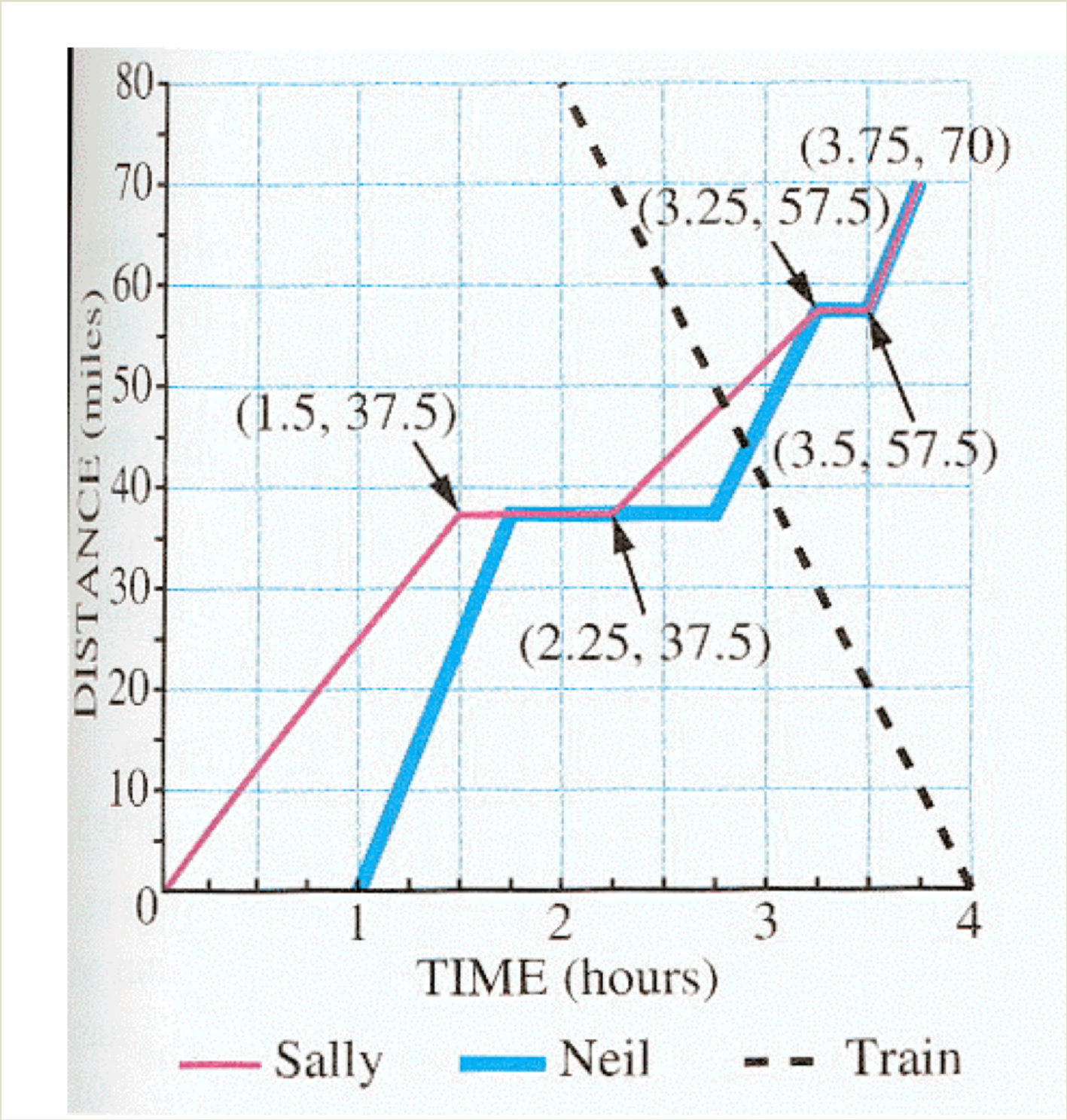
◊ "Real world" → equations and functions (not the other way around)

◊ Hands-on visual models (manipulatives and software)

“Real World”: The Bicycle Trip



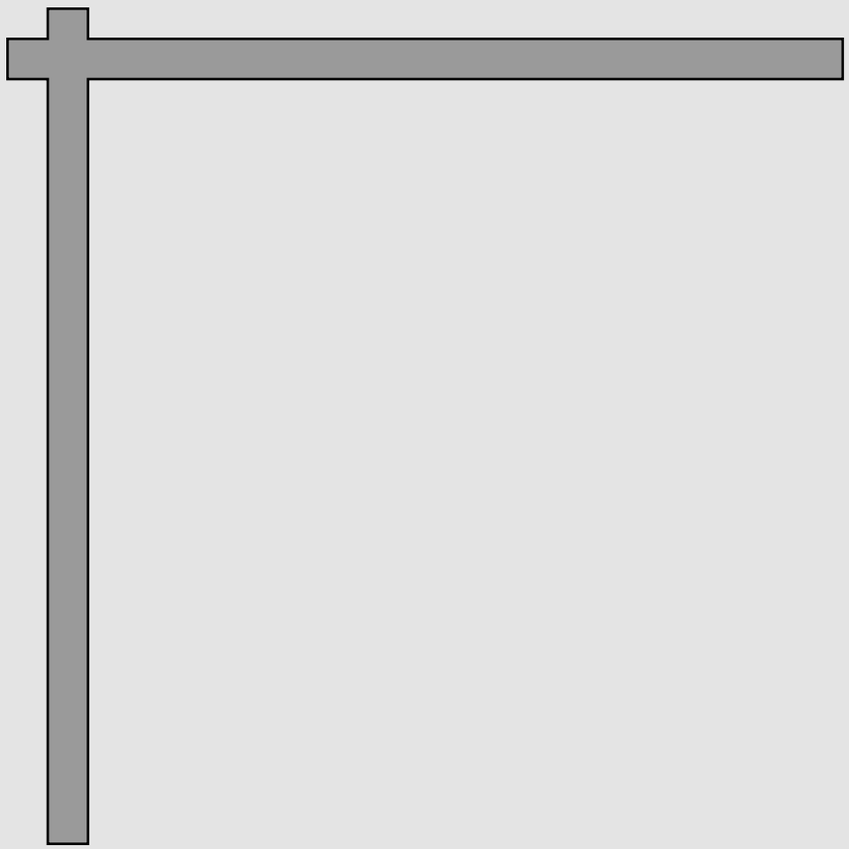
“Real World”: The Bicycle Trip



Hands-on: Completing the Square

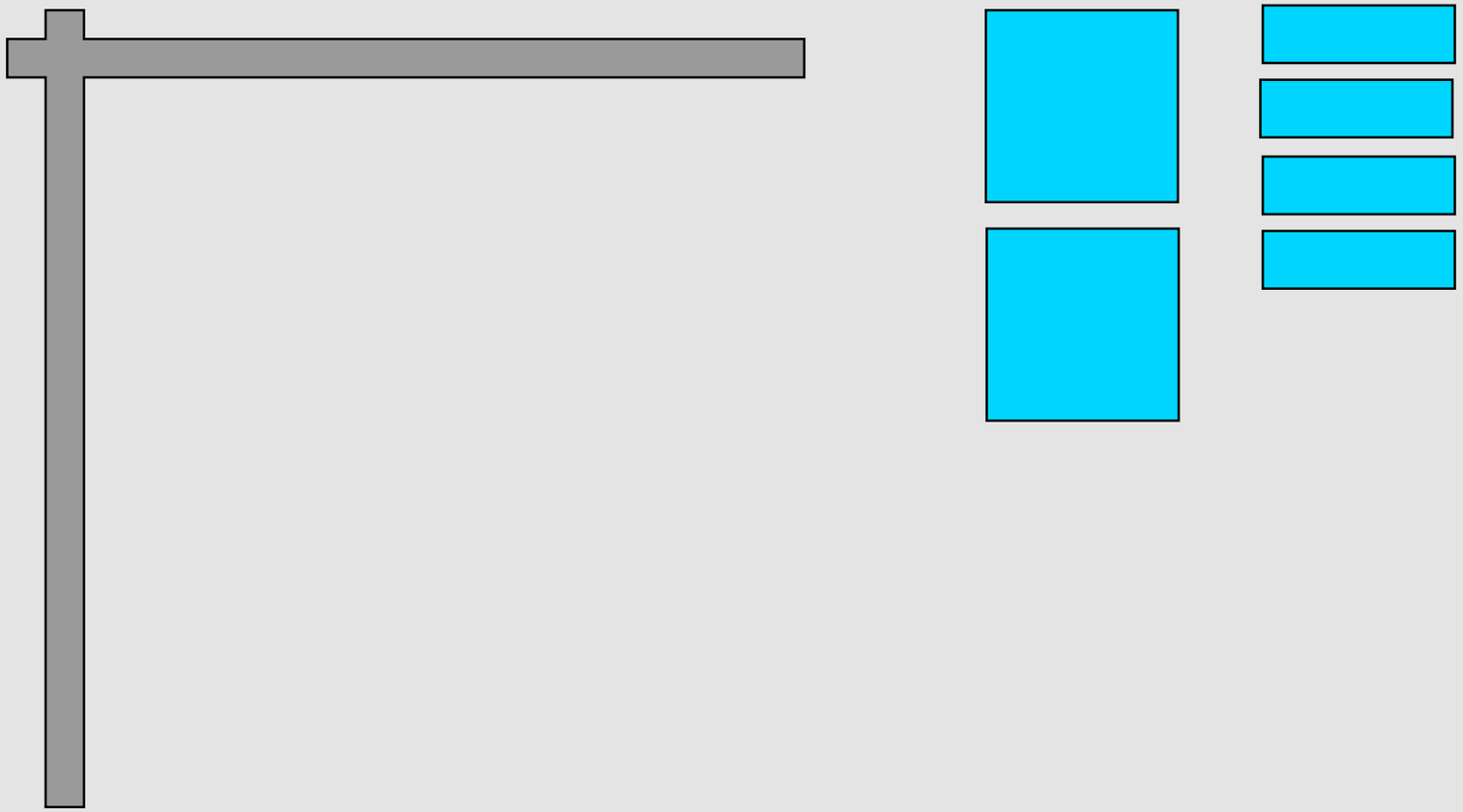
Make a Rectangle

$$2x^2 + 4x$$



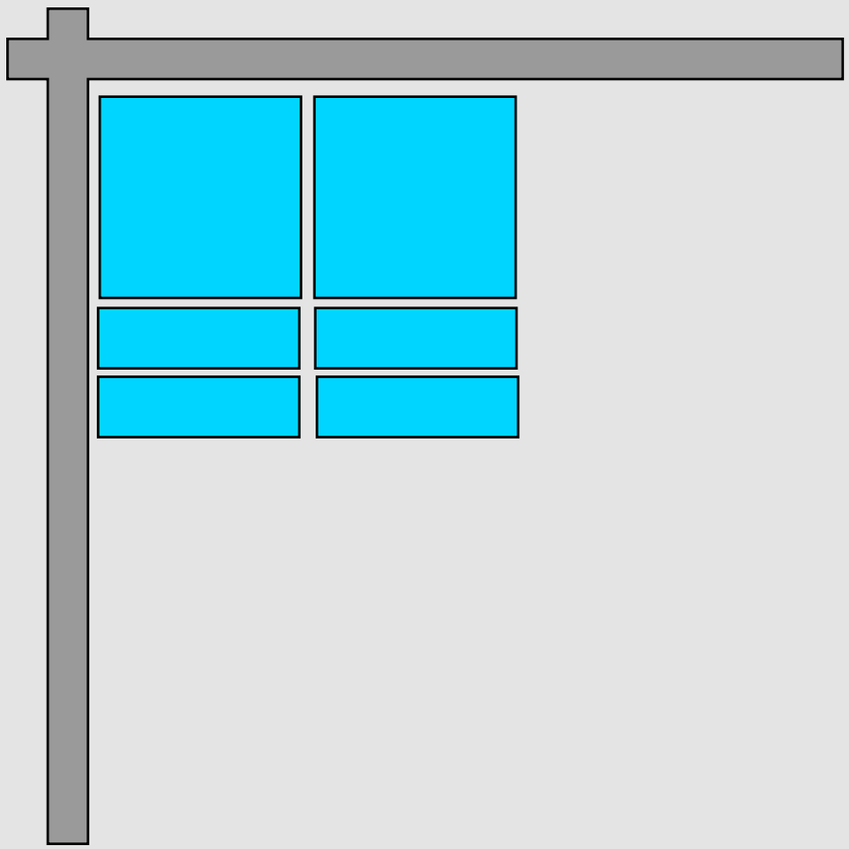
Make a Rectangle

$$2x^2 + 4x$$

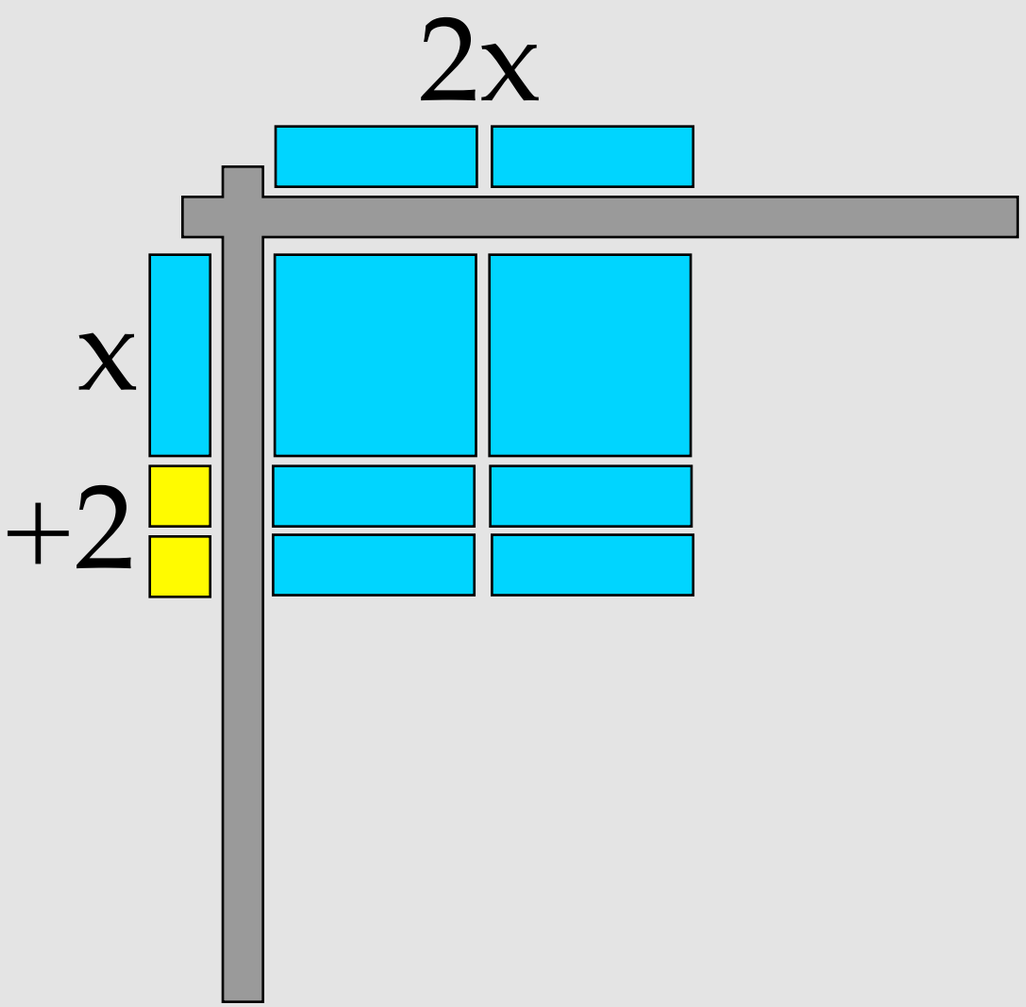


Make a Rectangle

$$2x^2 + 4x$$



Make a Rectangle

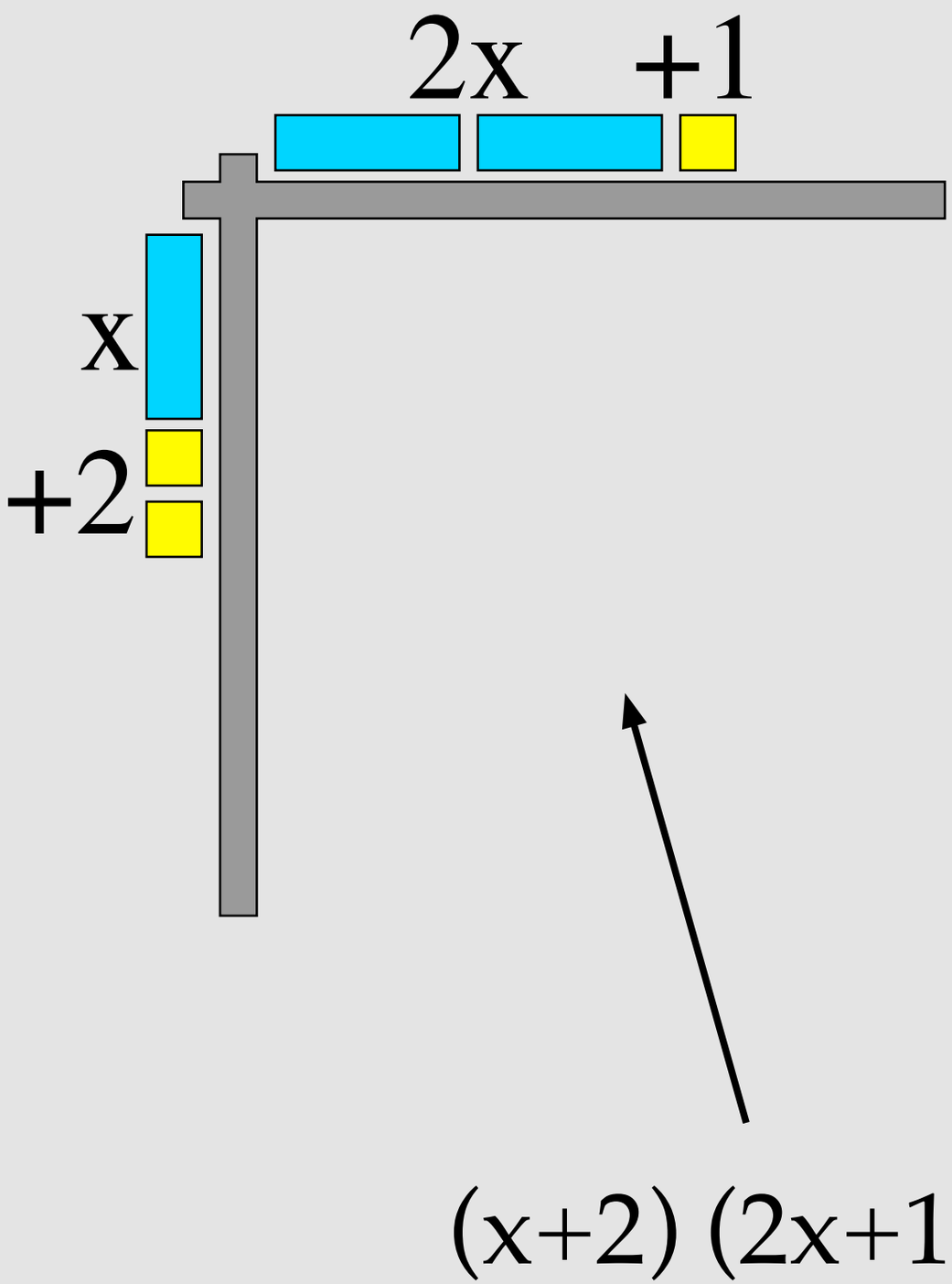


$$2x^2 + 4x = 2x \cdot (x+2)$$

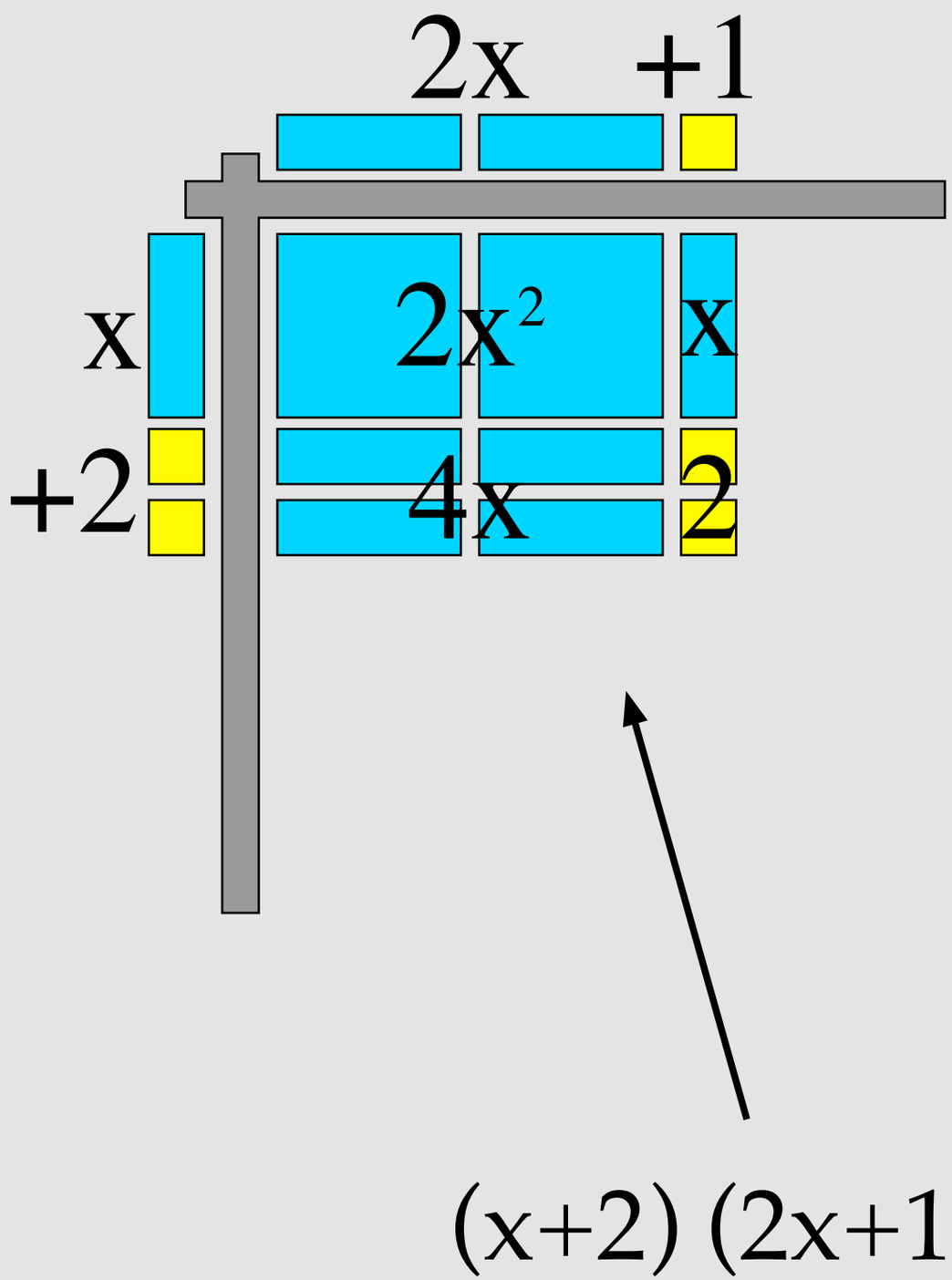
From blocks to symbols

$$(x+2)\,(2x+1)$$

From blocks to symbols

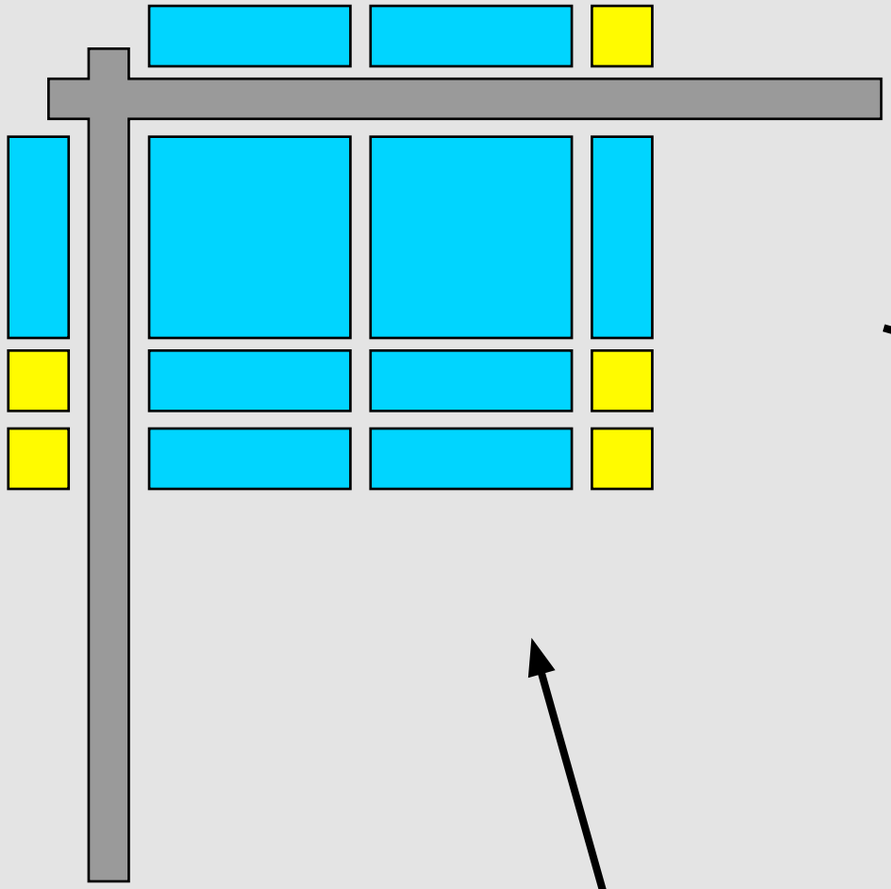


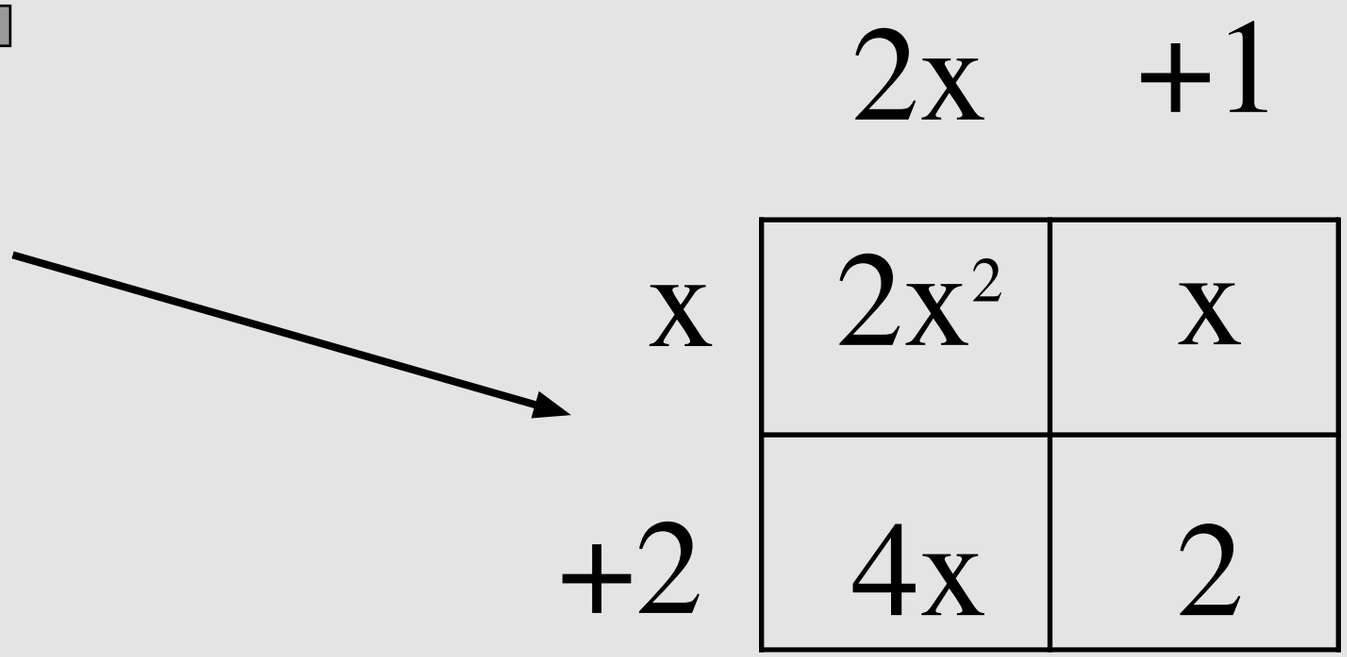
From blocks to symbols



$(x+2)\,(2x+1)$

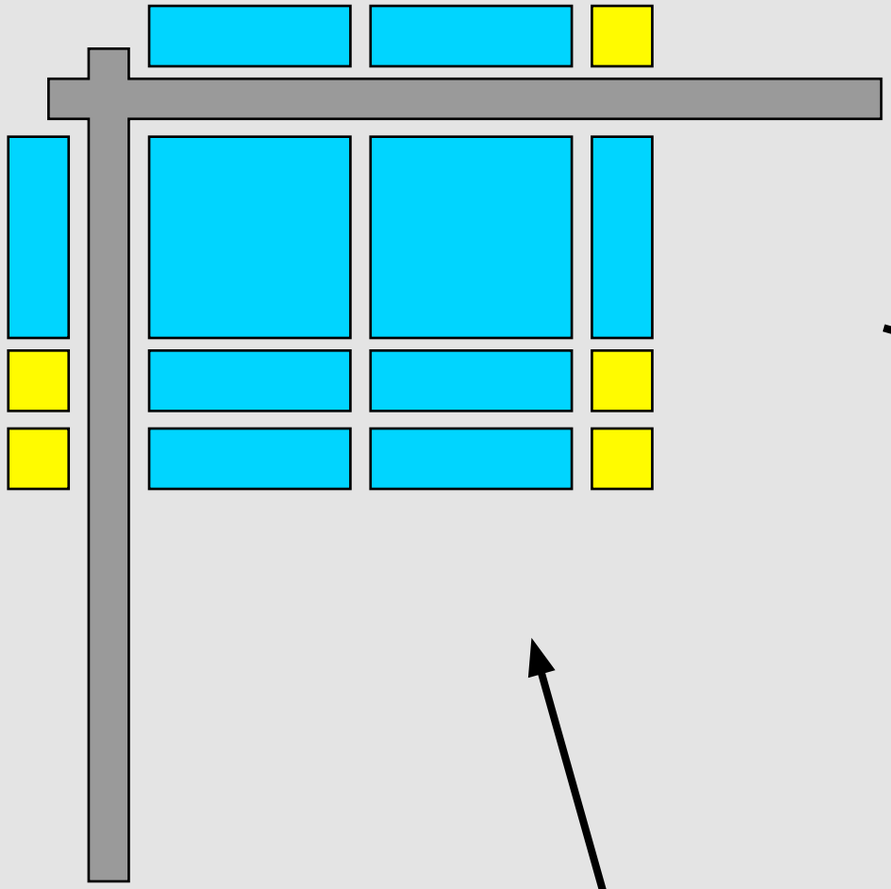
From blocks to symbols





(x+2) (2x+1)

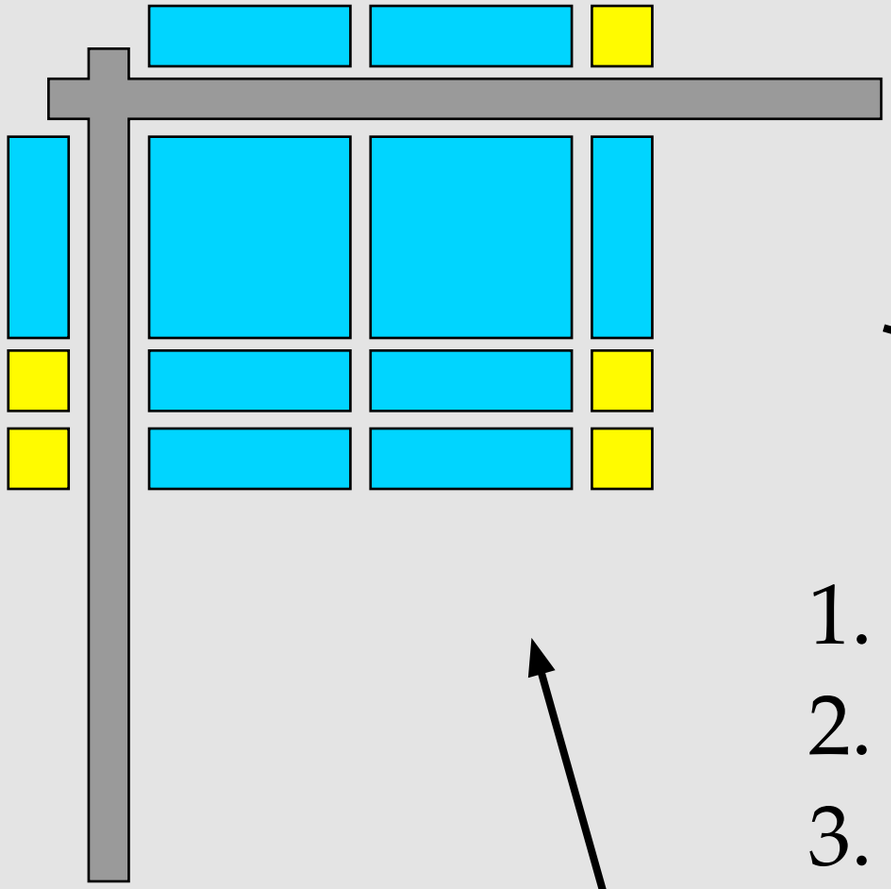
From blocks to symbols



	2x	+1
x	$2x^2$	x
+2	4x	2

$$(x+2)\,(2x+1) \quad = 2x^2 + 5x + 2$$

From blocks to symbols



1. Build
2. Sketch
3. “The Box”

	$2x$	$+1$
x	$2x^2$	x
$+2$	$4x$	2

$$(x+2)\,(2x+1) \quad = 2x^2 + 5x + 2$$

From blocks to symbols

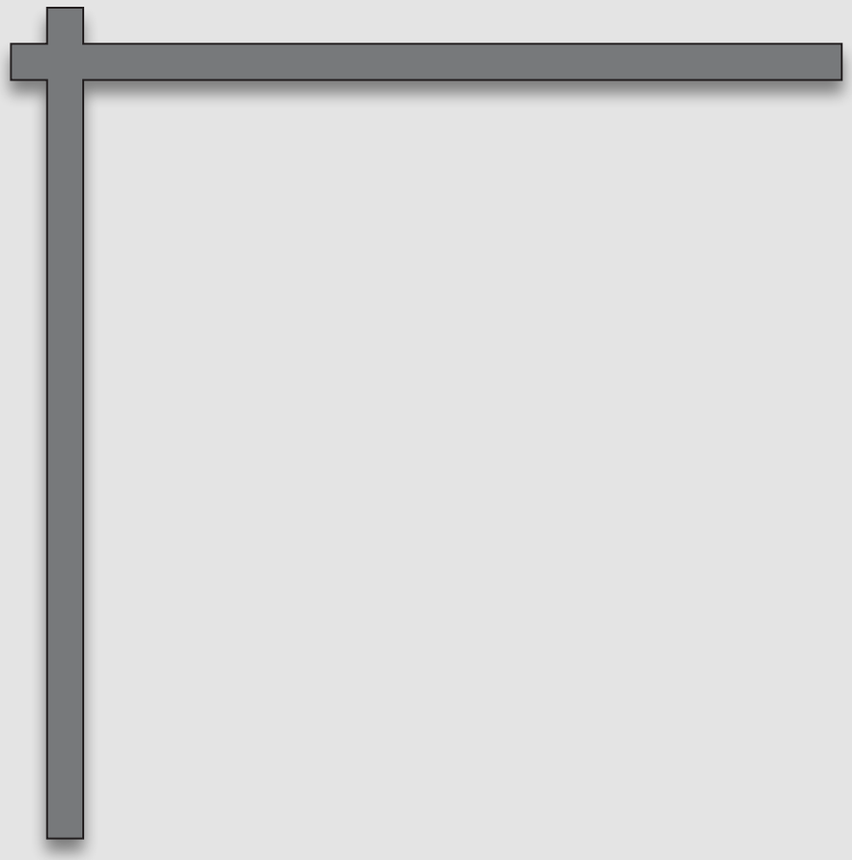
1. Build
2. Sketch
3. "The Box"

	$2x$	$+1$
x	$2x^2$	x
$+2$	$4x$	2

$$(x+2)\,(2x+1) \quad = 2x^2 + 5x + 2$$

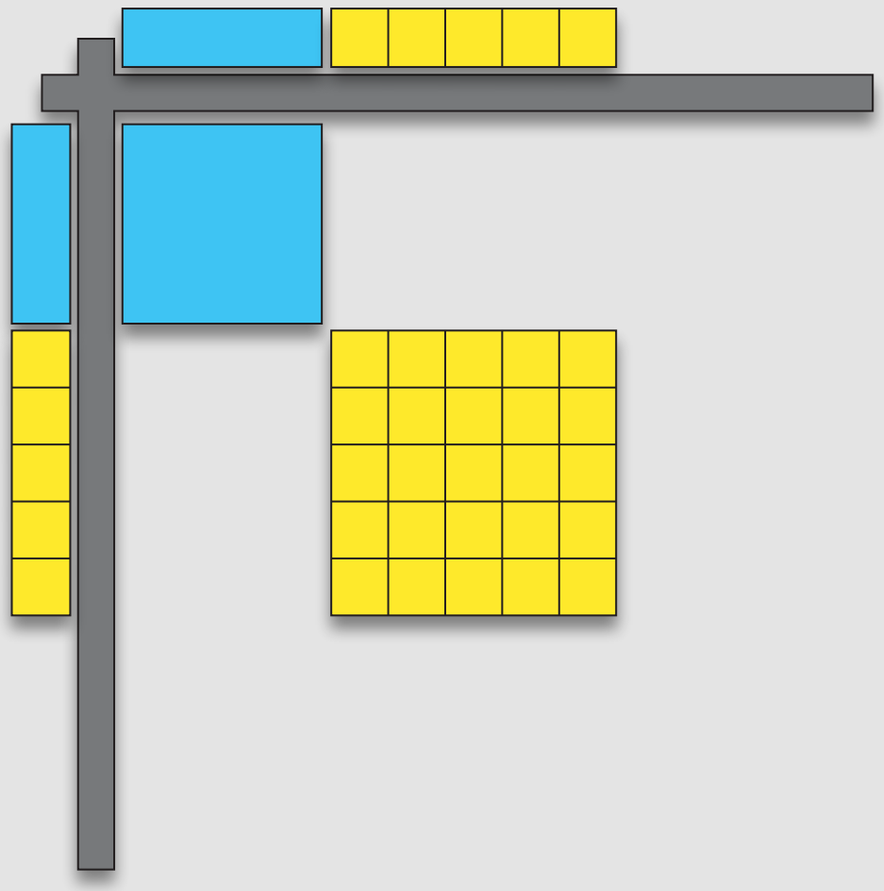
$$(x+5)^2 \stackrel{?}{=} x^2+25$$

"Distributing the square?"



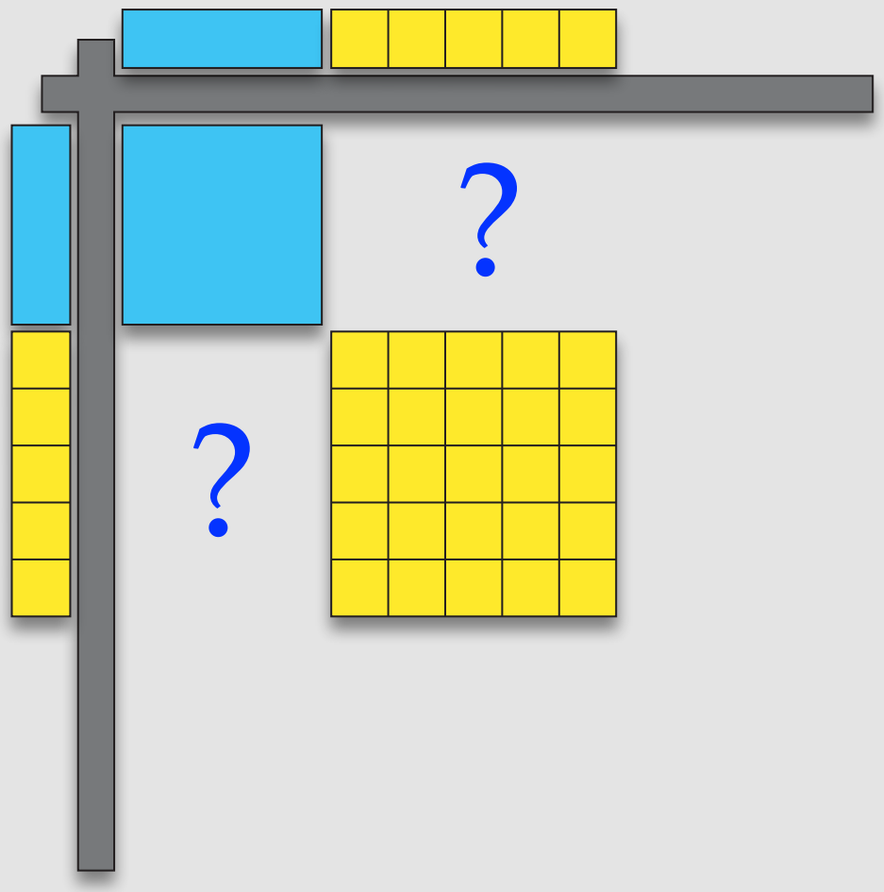
$$(x+5)^2 \stackrel{?}{=} x^2+25$$

"Distributing the square?"



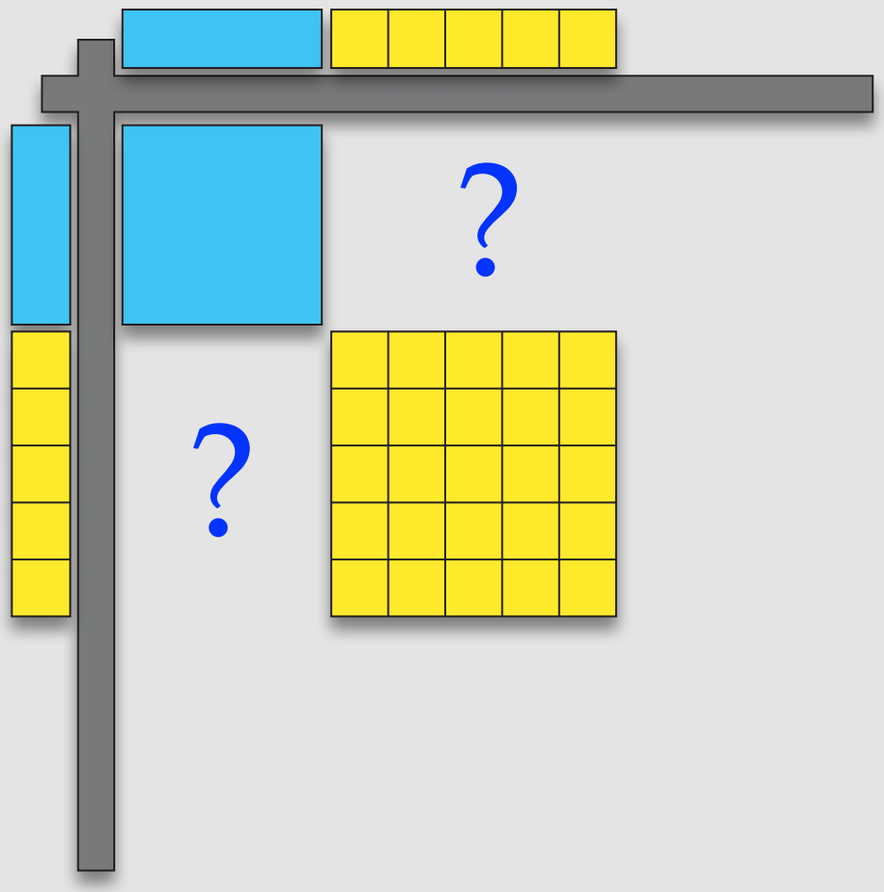
$$(x+5)^2 \stackrel{?}{=} x^2+25$$

"Distributing the square?"



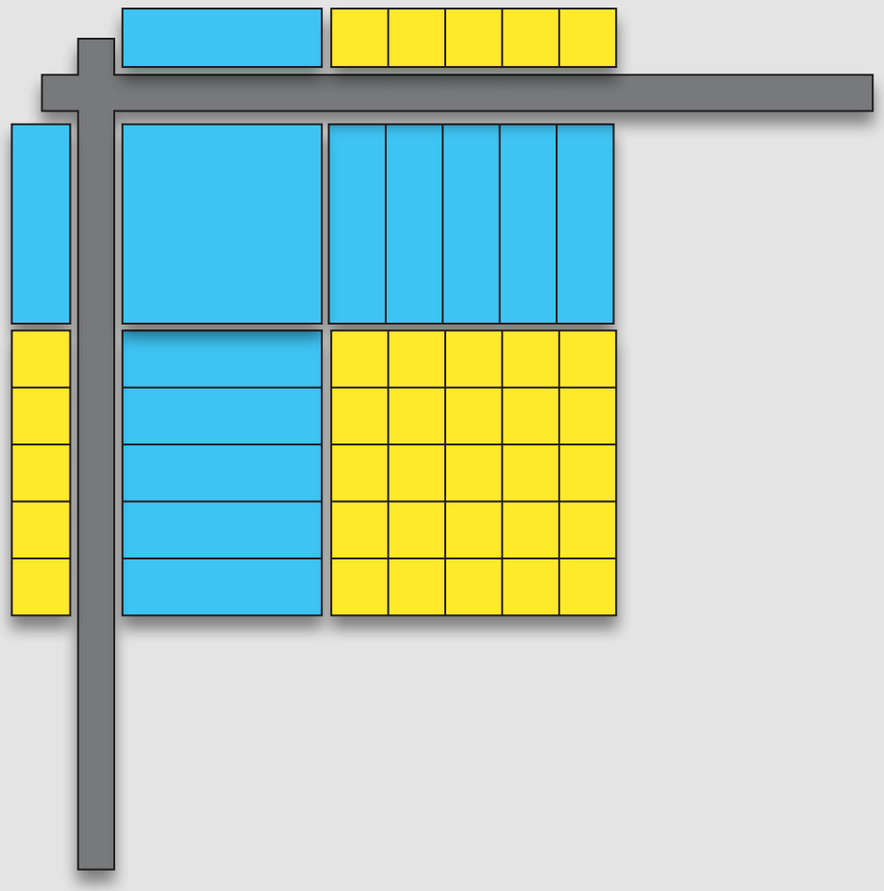
$$(x+5)^2 \neq x^2+25$$

"Distributing the square?"

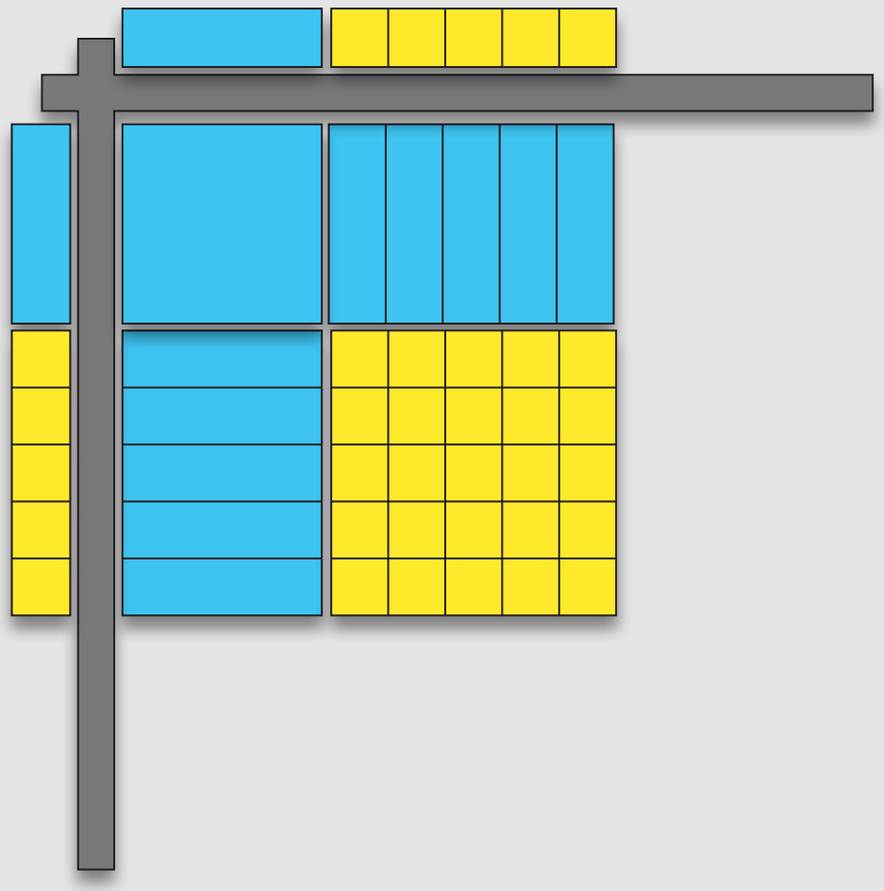


$$(x+5)^2 =$$

$$(x+5)^2 =$$



$$(x+5)^2 = x^2 + 10x + 25$$

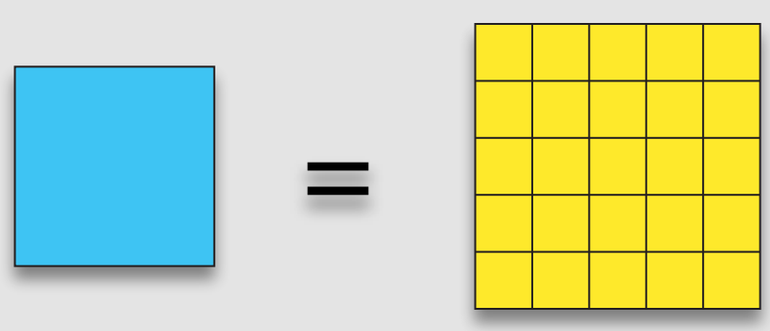


Equal Squares

$$x^2 = 25$$

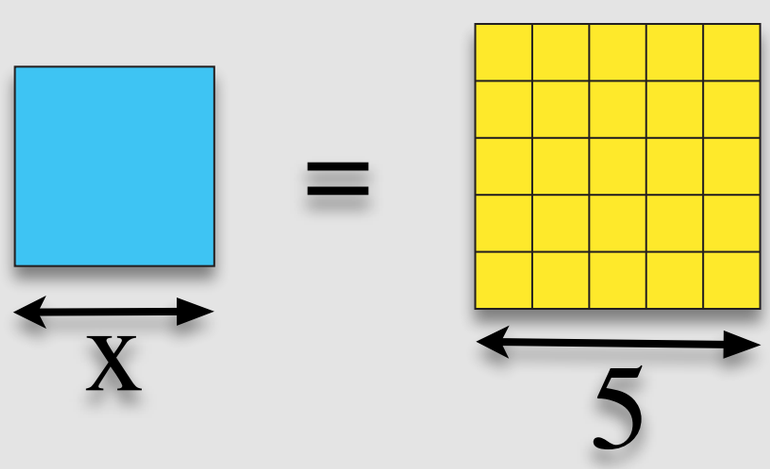
Equal Squares

$$x^2 = 25$$



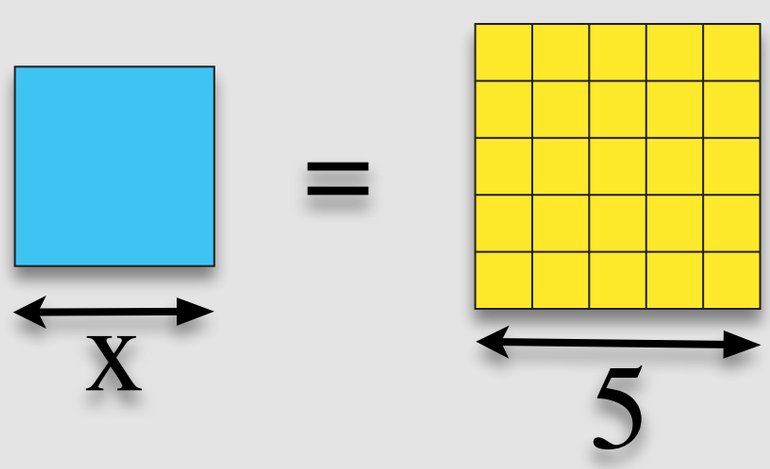
Equal Squares

$$x^2 = 25$$



Equal Squares

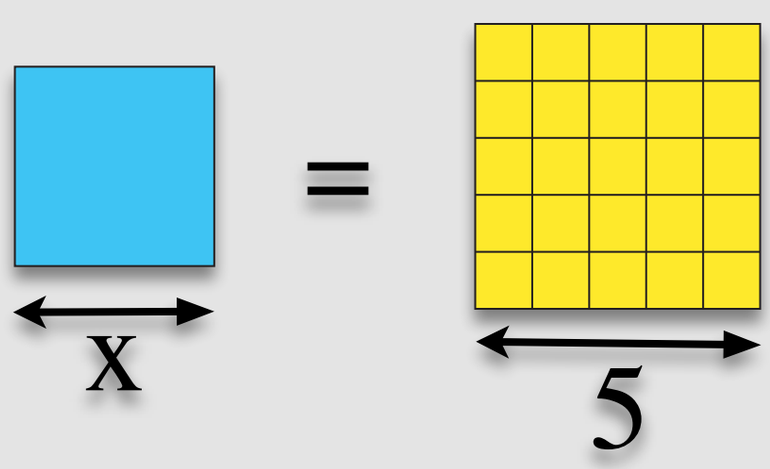
$$x^2 = 25$$



$$x = 5$$

Equal Squares

$$x^2 = 25$$



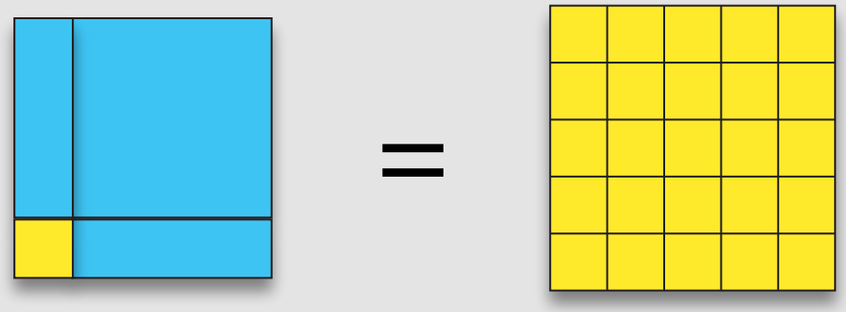
$$x = 5$$

or

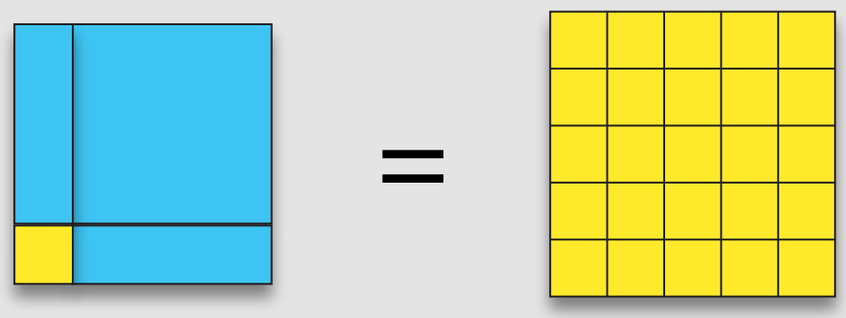
$$x = -5$$

$$x^2 + 2x + 1 = 25$$

$$x^2 + 2x + 1 = 25$$



$$x^2 + 2x + 1 = 25$$

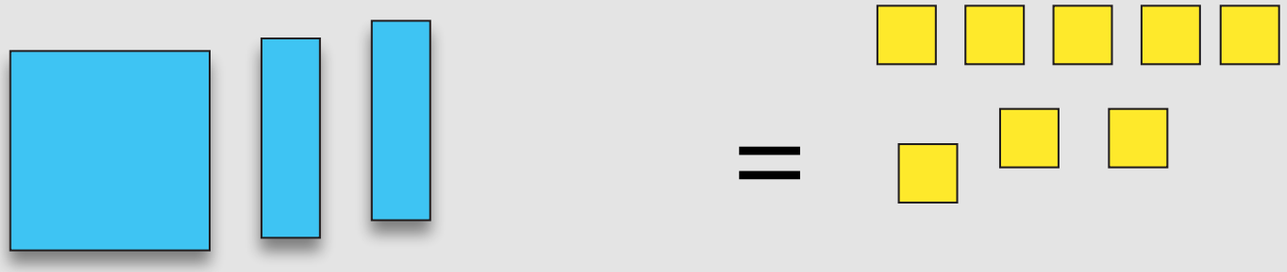


$$x + 1 = 5$$

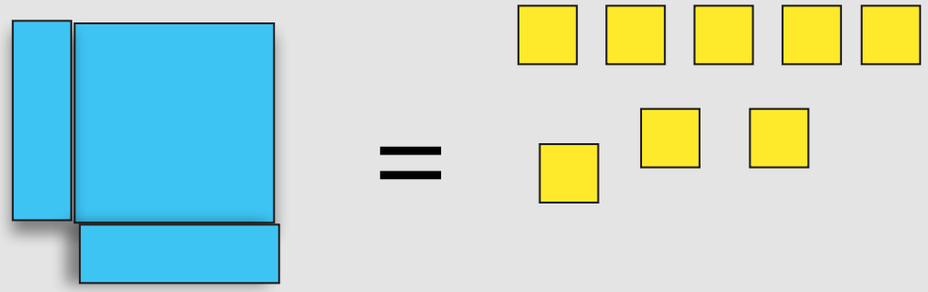
or

$$x + 1 = -5$$

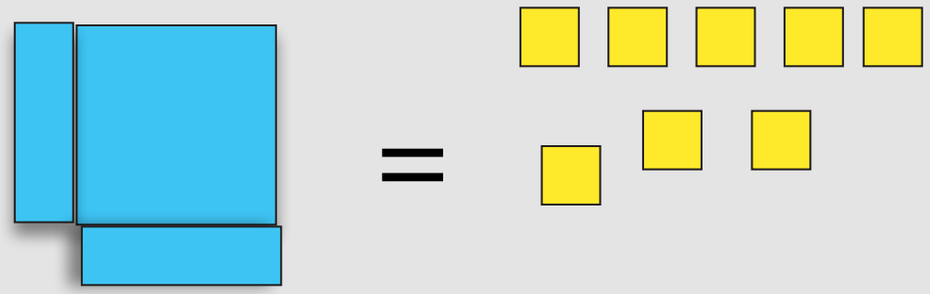
$$x^2 + 2x = 8$$



$$x^2 + 2x = 8$$

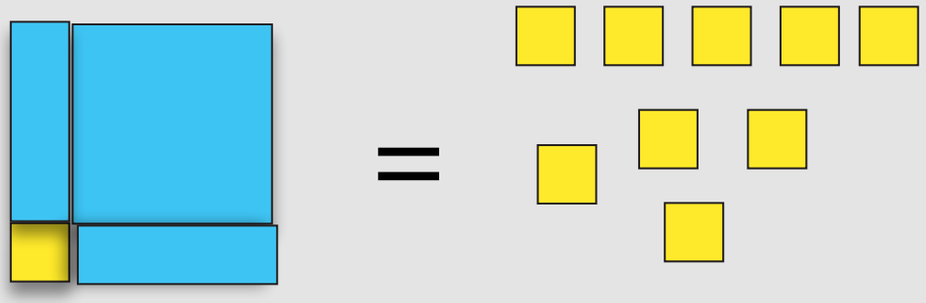


$$x^2 + 2x = 8$$



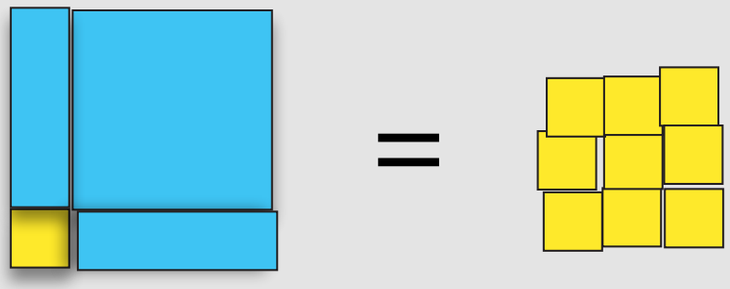
Complete the square

$$x^2 + 2x = 8$$



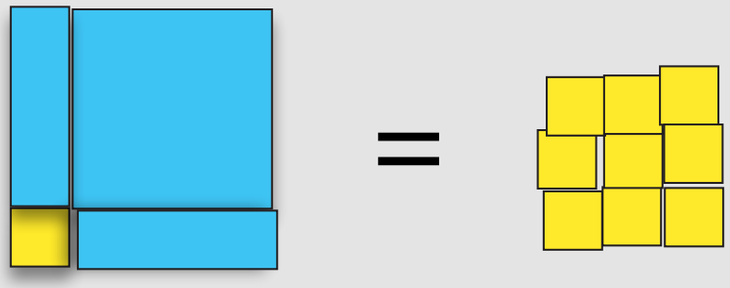
Complete the square

$$x^2 + 2x = 8$$



Complete the square

$$x^2 + 2x = 8$$



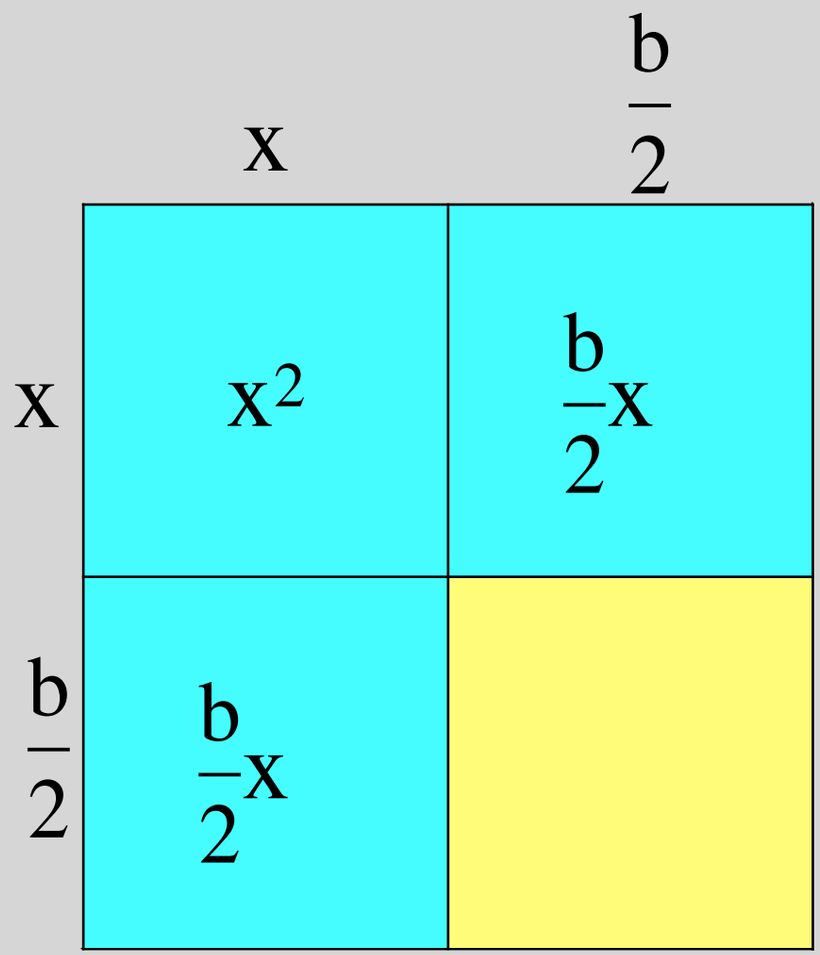
Complete the square

Symbolic Version

$$x^2 + bx + ?$$

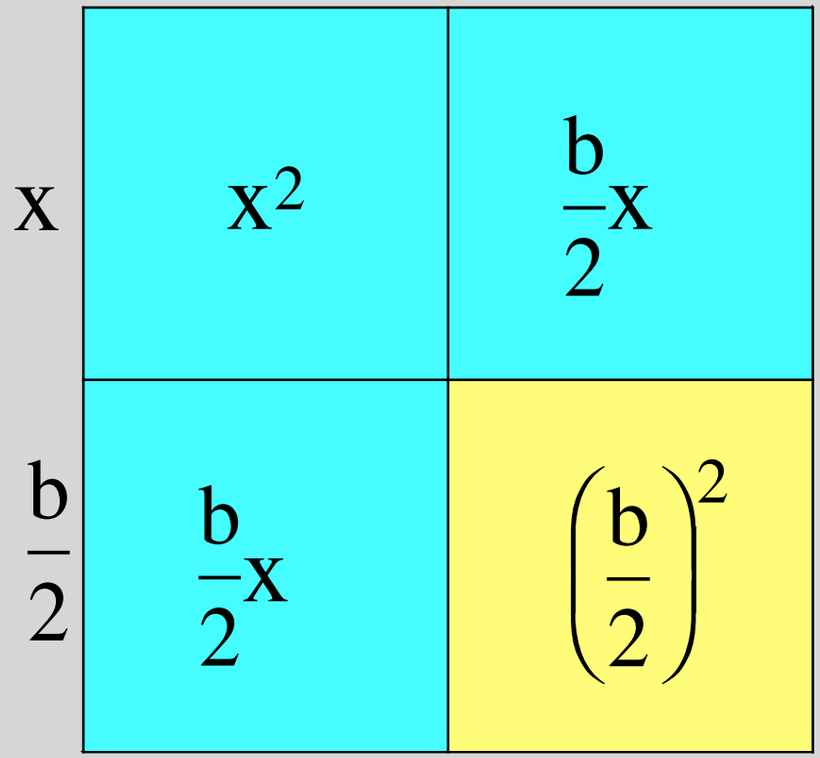
Symbolic Version

$$x^2 + bx + ?$$



Symbolic Version

$$x^2 + bx + ?$$



Algebra Strategy

Formulas should encapsulate understanding, not substitute for it.

◊ Go from concrete to abstract

◊ Teach the most important topics in more than one way

Geometry

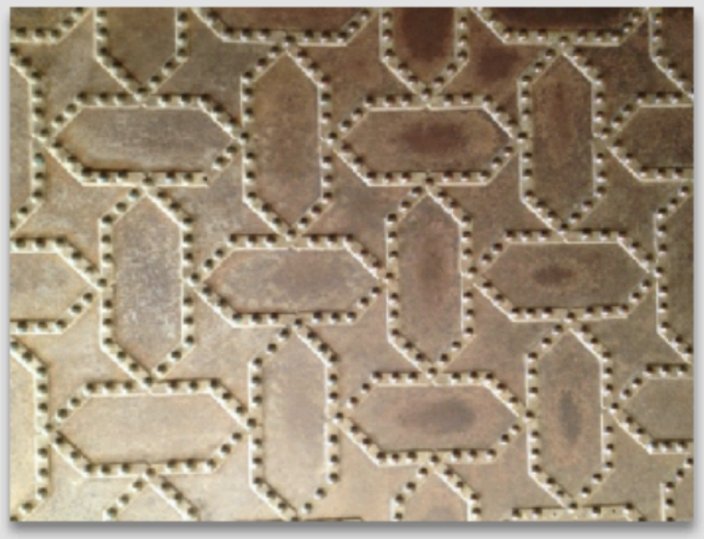
- The shrinkage of geometry has been a very long-term trend, largely due to the increased emphasis on other topics:
  - preparation for high school calculus
  - introduction of data and statistics
- Some of that is inevitable, but we need to be careful!

◊ Geometry is more popular than algebra.

◊ It has more aesthetic appeal.

◊ It links to the world in more obvious ways.

Enrichment: tiling and symmetry link to art in just about every culture worldwide.



Islamic Art

Escher

Theorem 2-1 *Midpoint Theorem*

If M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$ and $MB = \frac{1}{2}AB$.

Given: M is the midpoint of $\overline{AB}$.

Prove: $AM = \frac{1}{2}AB$; $MB = \frac{1}{2}AB$

Proof:

Statements	Reasons
1. M is the midpoint of $\overline{AB}$.	1. Given
2. $\overline{AM} \cong \overline{MB}$, or $AM = MB$	2. Definition of midpoint
3. $AM + MB = AB$	3. Segment Addition Postulate
4. $AM + AM = AB$, or $2AM = AB$	4. Substitution Prop. (Steps 2 and 3)
5. $AM = \frac{1}{2}AB$	5. Division Prop. of =
6. $MB = \frac{1}{2}AB$	6. Substitution Prop. (Steps 2 and 5)

Theorem 2-1 *Midpoint Theorem*

If M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$ and $MB = \frac{1}{2}AB$.

Given: M is the midpoint of $\overline{AB}$.

Prove: $AM = \frac{1}{2}AB$; $MB = \frac{1}{2}AB$

Proof:

Statements	Reasons
1. M is the midpoint of $\overline{AB}$.	1. Given
2. $\overline{AM} \cong \overline{MB}$, or $AM = MB$	2. Definition of midpoint
3. $AM + MB = AB$	3. Segment Addition Postulate
4. $AM + AM = AB$, or $2AM = AB$	4. Substitution Prop. (Steps 2 and 3)
5. $AM = \frac{1}{2}AB$	5. Division Prop. of =
6. $MB = \frac{1}{2}AB$	6. Substitution Prop. (Steps 2 and 5)

Geometry Strategy

◊ ***Introduce important topics with hands-on materials***

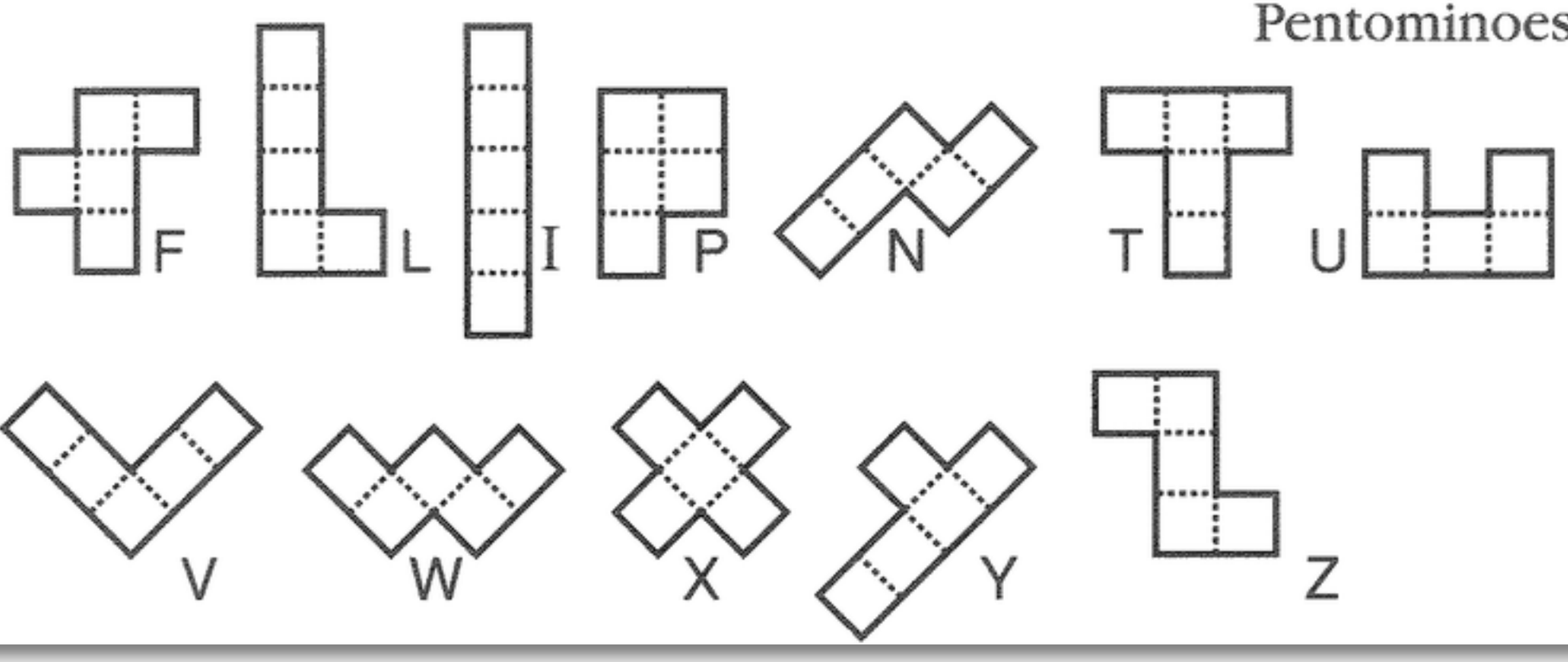
- Angles and polygons with tiling challenges
- The Pythagorean theorem with the geoboard
- Construction with GeoGebra
- Similarity with geometric puzzles

◊ Build proof on that foundation

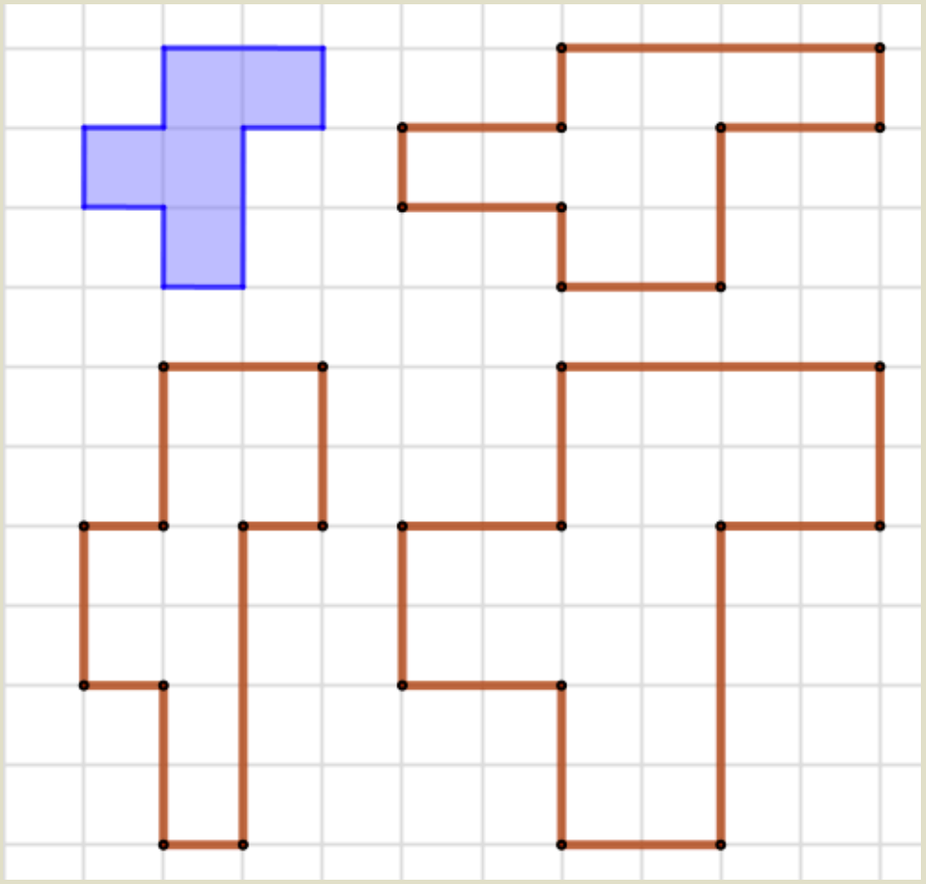
- ***Proof is to dispel doubt, not display compliance***
- Prioritize logic and understanding over format and notation

Example: Similarity

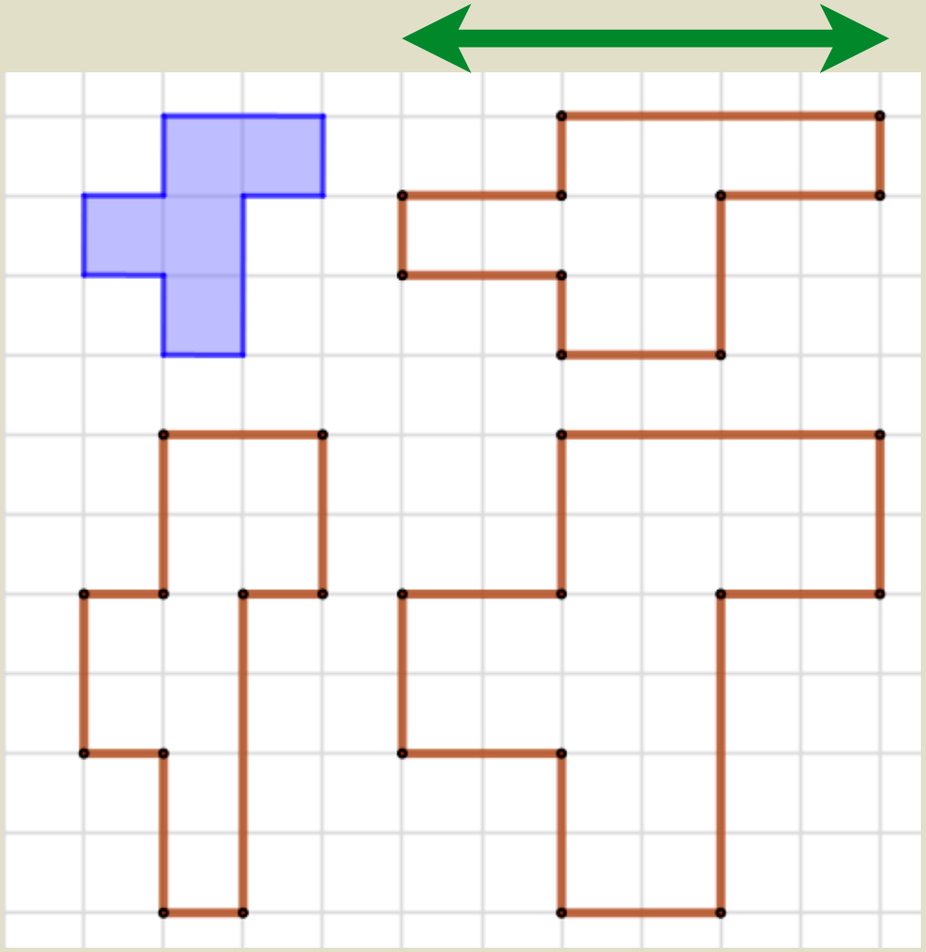
Pentominoes



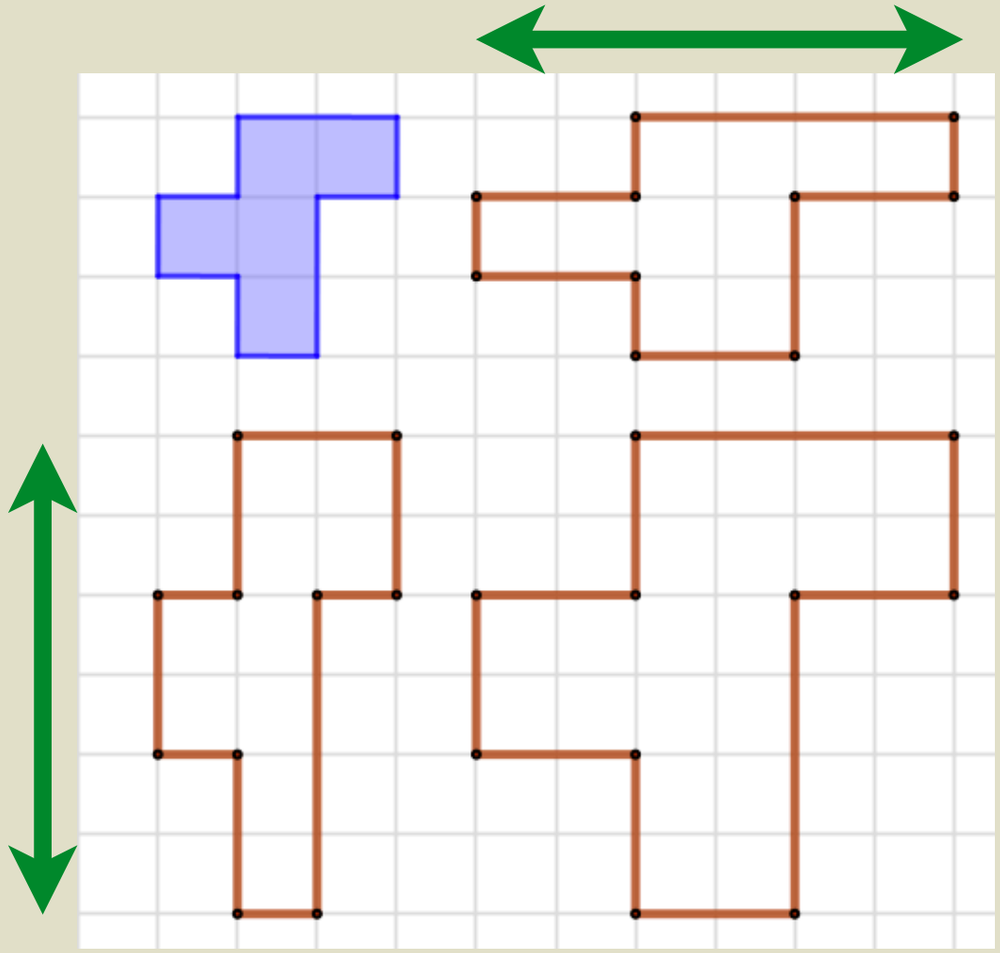
Proportions



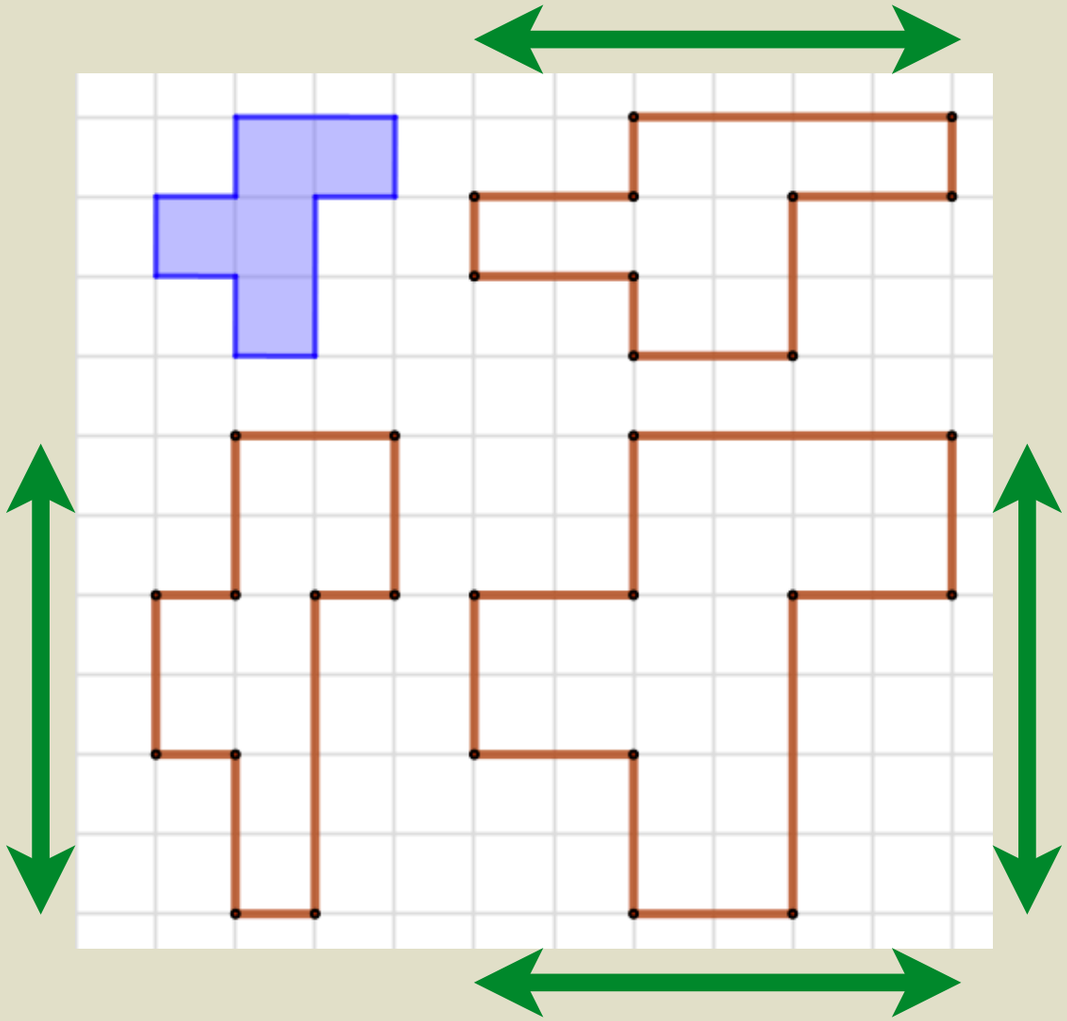
Proportions



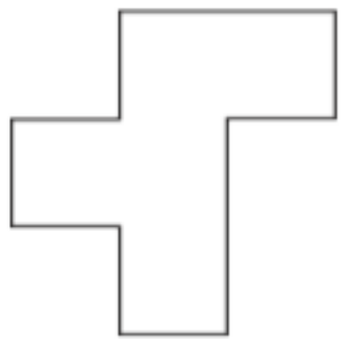
Proportions



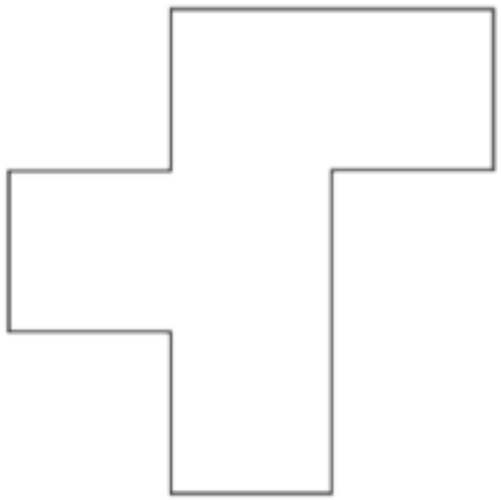
Proportions



Pentomino Blowups

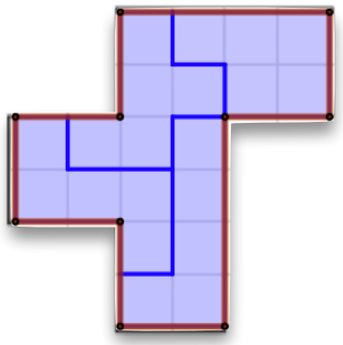


double the dimensions

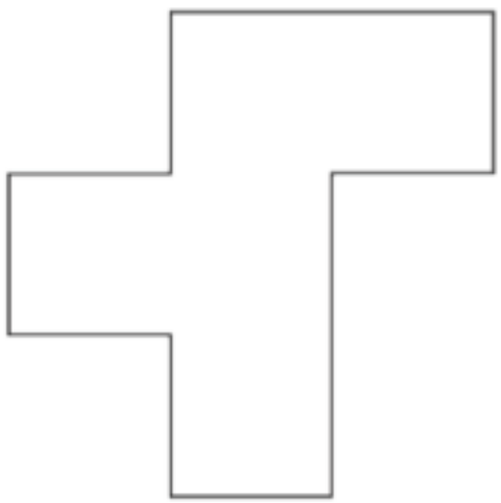


triple the dimensions

Pentomino Blowups



double the dimensions



triple the dimensions

Conclusion:
Algebra and Geometry for All Students!

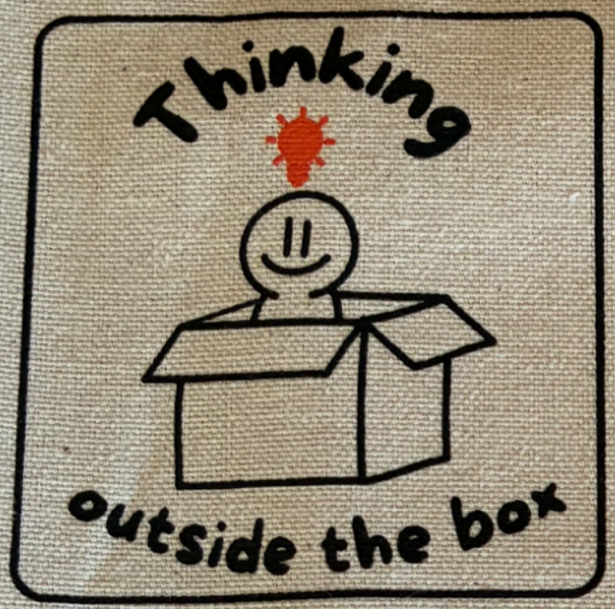
◊ Reading and writing are more important, but ***math is a literacy***, necessary for further work in all STEM fields.

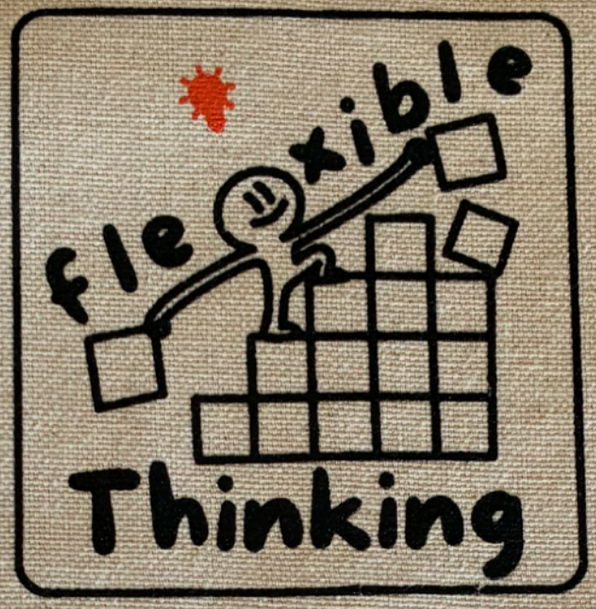
◊ It helps develop ***habits of mind*** that pay off no matter where a student's life might lead them

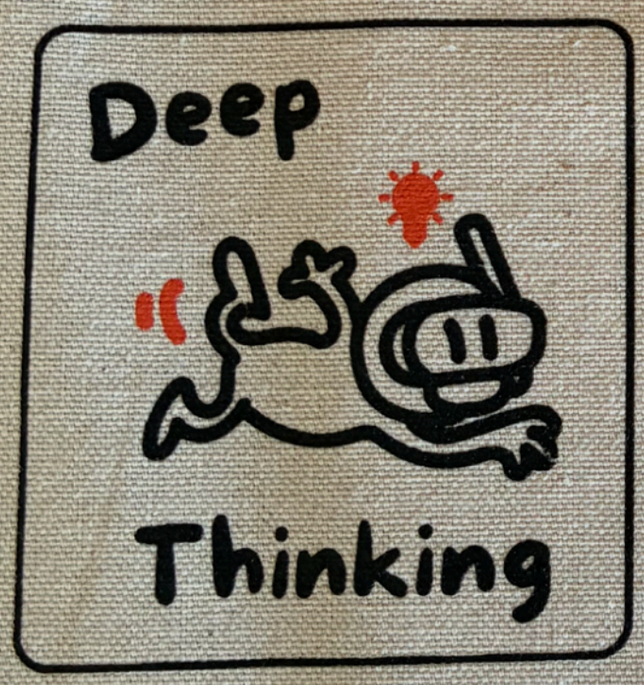
For All Students:
Habits of Mind

- ◊ Concentration
- ◊ Perseverance
- ◊ Mental arithmetic
- ◊ Abstraction
- ◊ Ownership
- ◊ Trying things
- ◊ Initiative
- ◊ Problem posing
- ◊ Argumentation
- ◊ Precision in
  - Speaking
  - Writing

Math makes you







Think!

Innovamat

Interested in STEM now

May be interested in STEM someday

All students

Interested in STEM now

May be interested in STEM someday

Conference and Speaker Evaluation

https://tinyurl.com/CMCNEVAL

Henri Picciotto

henri@MathEducation.page

www.MathEducation.page/talks.html →

blog.MathEducation.page

BlueSky, Mathstodon, Twitter: @hpicciotto