

Isometries of the Plane Teacher's Notes

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This unit is intended to be consistent with the Common Core State Standards for Mathematics (CCSSM), but it does go quite a bit further than is required by the Standards. It can be used in the professional development of teachers, to give them more depth of understanding of the subject, or in a fourth-year high school course. (An example of such a course is *Space*, which I taught in various versions between 1991 and 2013. See www.MathEducationPage.org/space. I developed most of this unit for that class.)

Isometries are the transformations of the plane that preserve distance. (In the CCSSM, they are called *rigid transformations*, and are also assumed to preserve angles.) This unit is primarily about isometries, but it will also touch on dilations, both because they are a key CCSSM topic, and because they provide a worthwhile “non-example” of isometry.

The unit includes a number of references to activities in GeoGebra. They can readily be adapted to other dynamic geometry environments.

See the **Transformational Geometry** page on my Web site for crucial related materials, including some geared to grades 8-10: www.MathEducationPage.org/transformations

Simulating the Transformation Tools

This activity (suitable for grades 8-12) assumes that the student has a basic familiarity with GeoGebra, including the main transformation tools. The purpose of the lesson is to help the student formulate precise definitions of the main transformations, in the style suggested by the CCSSM.

Glide Reflection

A glide reflection is the composition of a reflection with a translation in a vector parallel to the line of reflection. It is not mentioned in the CCSSM, but it is essential to this unit, and to secondary-school-appropriate work on symmetry.

Isometry Specifications

This lesson helps students develop their geometric understanding of the isometries.

A rigorous answer to #4 requires some background in geometry, for example some familiarity with dilations, or with triangle similarity.

Recognizing Isometries

This lesson reviews the isometries with an emphasis on them as functions.

Composing Transformations

#1-4 are by far the most important problems in this lesson, because they help us establish results we will need subsequently.

How Many Points?

Three Points

The first of these lessons provides a preview of the crucial theorem we prove in the second one.

Three Reflections Suffice

The proof is straightforward if students have experience with geometric construction. However they may have trouble understanding that because of the theorem proved in the previous lesson, the argument about a triangle and its image is enough to fully answer this question for all points.

Note that the argument is essentially the same as the proof of SSS in “Triangle Congruence and Similarity”. You can find two versions of this article on my Transformational Geometry page. (See link above.)

One, Two, or Three Reflections

This lesson provides a review of **Composing Transformations** #1-4, and launches the proof that any isometry is one of these four: reflection, translation, rotation, and glide reflection. The next lesson completes the proof.

Three Reflections: Six Cases

Experimenting in an interactive geometry environment should yield conjectures for #1-2. Proving these results and figuring out the rest of the page will require teacher help.

One way to solve #3c and #4 is to reduce each problem to case #3b.

The whole proof is spelled out on my Web site:

<https://www.mathedpage.org/transformations/isometries/four>

That presentation, and the applets therein, can serve as a guide to classroom exploration and discussion.

Acknowledgments

Some ideas in this unit were inspired by Richard Brown's excellent *Transformational Geometry*. Of course, since the book came out in 1973, it predates not only the Common Core, but also interactive geometry software, so it should not be used as is. Also note that in his proof of the final result — that every isometry is one of translation, rotation, reflection, or glide reflection— Brown fails to address all the cases.

Thanks to Zalman Usiskin for his feedback on this unit. He recommends these books:

Zalman Usiskin, Anthony Peressini, Elana Marchisotto, and Dick Stanley, *Mathematics for High School Teachers: An Advanced Perspective*.

Barker, W., & Howe, R. (2007). *Continuous symmetry*. Providence, RI: American Mathematical Society

Art Coxford and Zalman Usiskin *Geometry – A Transformation Approach*

Simulating the Transformation Tools

In this GeoGebra activity, you will construct the images of a scalene triangle under each of the main geometric transformations, *without using the transformation tools*. This will help you get a strong understanding of the transformations, and prepare you to write a precise definition for each.

Reflection

1. **Given:** a line b and a scalene $\triangle ABC$.
Construct: $\triangle A'B'C'$, the reflection of $\triangle ABC$ in b .
Suggested tools: Perpendicular Line, Circle with Center through Point.
When you are done, you may hide the intermediary steps in the construction.
2. Check the correctness of your construction by moving the line, by moving the pre-image, and finally by using the Reflection tool.
3. Write a precise definition of the reflection of a point P in a line b . Don't forget to mention the case where P is on b .

Translation

4. **Given:** a vector u and a scalene $\triangle ABC$.
Construct: $\triangle A'B'C'$, the image of $\triangle ABC$ under a translation by vector u .
Suggested tools: Parallel Line, Compass.
When you are done, you may hide the intermediary steps in the construction.
5. Check the correctness of your construction by changing the vector, by moving the pre-image, and finally by using the Translation tool.
6. Write a precise definition of the image of a point P under a translation by vector u .

Rotation

7. **Given:** a point O , an angle α and a scalene $\triangle ABC$.
Construct: $\triangle A'B'C'$, the image of $\triangle ABC$ under a rotation with center O and angle α .
Suggested tools: Ray, Circle, Angle with Given Size.
(It is possible to avoid using the latter, but it is much more difficult without it.)
When you are done, you may hide the intermediary steps in the construction.
8. Check the correctness of your construction by moving the center, moving the pre-image, and finally by using the Rotation tool.
9. Write a precise definition of the image of a point P under a rotation (center O , angle α .)

Dilation

10. **Given:** a point O , a number r and a scalene $\triangle ABC$.
Construct: $\triangle A'B'C'$, the image of $\triangle ABC$ under a dilation with center O and scaling factor r .
Suggested tools: Line, Distance or Length, Circle with Radius. (Optional: Slider)
Hint: the radius can be entered as a multiplication.
When you are done, you may hide the intermediary steps in the construction.
11. Check the correctness of your construction by moving the center, moving the pre-image, and finally by using the dilation tool.
12. Write a precise definition of the image of a point P under a dilation with center O and scaling factor r .

Glide Reflection

Consider these two polygons:

1. Find several ways to get from one to the other using exactly two isometries (translation, rotation, reflection.)
2. If you did #1 correctly, you found that every solution included one reflection. The other isometry was a translation or a rotation.
 - a. Explain why this had to be the case.
 - b. If you only found examples involving translations, find at least one involving a rotation, and vice-versa.

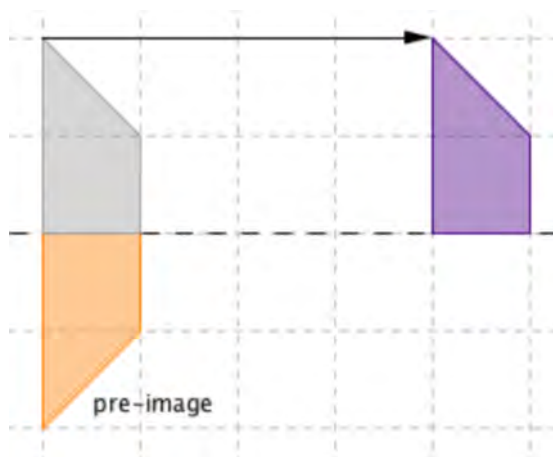


The sequence illustrated below is called a *glide reflection*:

The pre-image is reflected in the line, and the resulting polygon is translated by the vector. In a glide reflection, the translation vector must be parallel to the line of reflection.

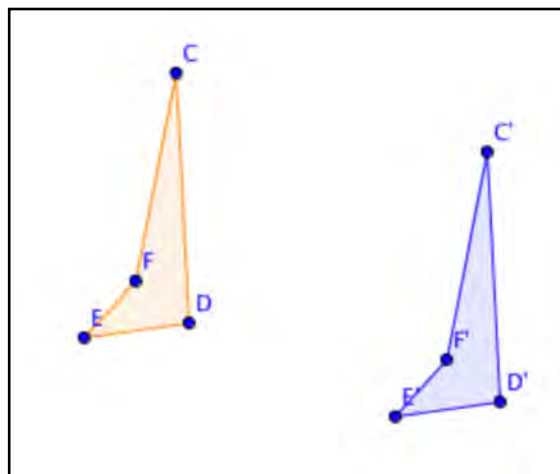
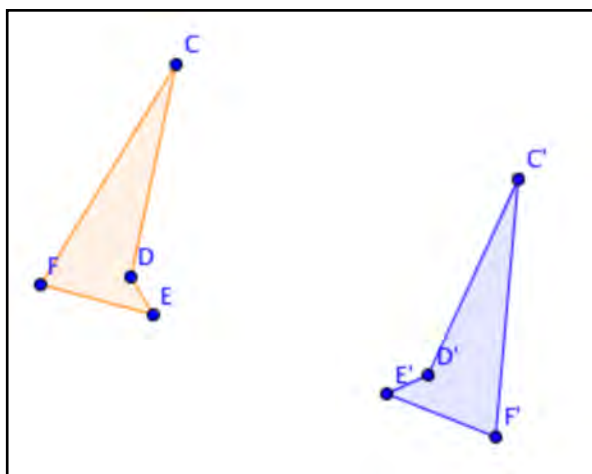
3. Draw the intermediate polygon if the translation is done first, then the reflection.

The fact that the two transformations can be done in either order is one reason to choose this particular sequence over all the others and give it a special name. Another reason is that glide reflections occur in nature and in art, as you can see in the example below, where the same vector and reflection line are used repeatedly. (Figure from *Handbook of Regular Patterns* by Peter Stevens, a remarkable resource.)



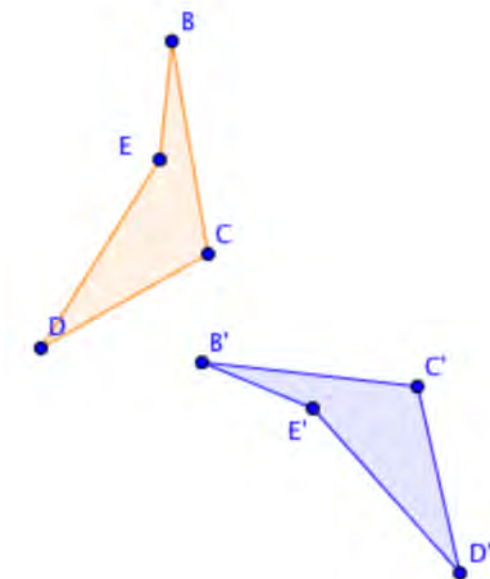
Isometry Specifications

One of these figures shows a reflection, the other shows a translation.

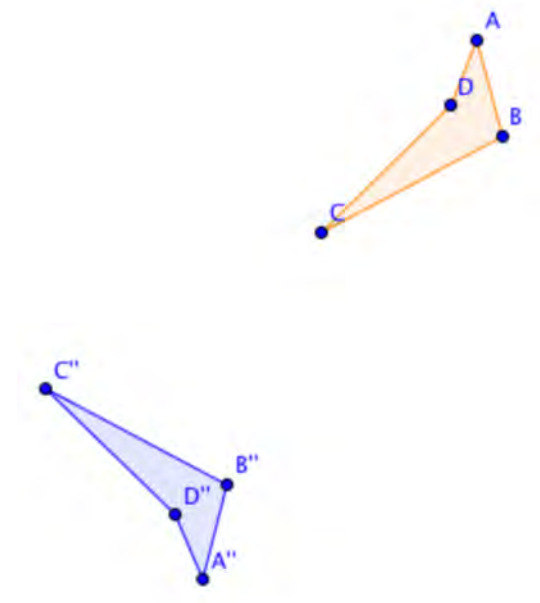


1.
 - a. Draw the line of reflection and the translation vector on the figures.
 - b. Explain in words how to find the line of reflection.
 - c. Explain in words how to find the translation vector.

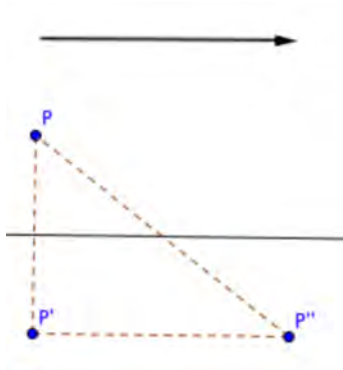
2. This figure shows a rotation. Find the center of rotation. **Hint:** is it closer to C or C'?



This figure shows a glide reflection. It is not obvious where the line of reflection is.



3. Study this figure, which shows one point P and its image P'' in a glide reflection. Mark any right angles and equal segments.



4. Find the line of reflection for the figure at the top of the page. **Hint:** in #3, where is the midpoint of PP'' ? Why?

Recognizing Isometries

Under the *identity mapping*, every point is its own image.

1. Is the identity mapping an isometry?

In answering #2-4, be specific. For example, do not just say “a rotation” — specify its center and angle, if possible. Likewise for other isometries.

2. Reflect the capital letter F in a line, and then reflect the image in a line parallel to the first one, with a distance d between the lines.
- Is orientation preserved by this process?
 - Is distance preserved?
 - How could you transform the letter into its final image in just one isometry?
3. Repeat, reflecting the F in two perpendicular lines.
4. In each case below, what isometry gives the following image for the triangle $\{(0,0), (0,2), (1,0)\}$?
- $\{(3,4), (3,6), (4,4)\}$
 - $\{(3,5), (3,7), (2,5)\}$
 - $\{(0,4), (2,4), (0,3)\}$

A *fixed point* for a given isometry is a point that is its own image.

5.
 - If an isometry has three non-collinear fixed points, what is the isometry?
 - If an isometry has more than one fixed point, and they are all collinear, what is the isometry?
 - If an isometry has exactly one fixed point, what is the isometry?
 - Which isometries have no fixed points?

An *invariant set* is a set of points that is left unchanged by a mapping. Individual points can move, but the set as a whole remains the same.

6. Find examples of invariant lines and circles for each of the four types of isometries.

Composing Transformations

Two Reflections

We will write Ref_a for “reflection in the line a .”

1. Consider two parallel lines, b and c . What is the composite of Ref_b and Ref_c ? Experiment on graph paper and/or GeoGebra and make a conjecture about this. **Important:** you need to describe the resulting transformation specifically, and in reference to lines b and c (not what particular pre-image you used when experimenting.)
2. Show that your conjecture is valid with the help of a diagram. **Hint:** remember the precise definition of reflection.
3.
 - a. If you execute the reflections in reverse order, is the resulting isometry the same?
 - b. If the composite of reflections in parallel lines d and e is exactly the same transformation you found in #1-2, what can you say about lines d and e ?

Definition: Consider two transformations M and N . The *composite* of M and N is a transformation that for all points P , maps P to $P' = N(M(P))$.

In other words, the transformations are executed in succession, first M , then N . One example we have already encountered is the glide reflection.

4. Now answer questions 1-3 if lines b and c are not parallel.
5.
 - a. Given a translation T , find two reflections whose composition is T .
 - b. Given a rotation R , find two reflections whose composition is R .

Other Compositions

The following are approximately in order from easiest to most challenging. Remember: you need to describe the resulting transformation specifically, and in reference to the specifics of the two original transformations.

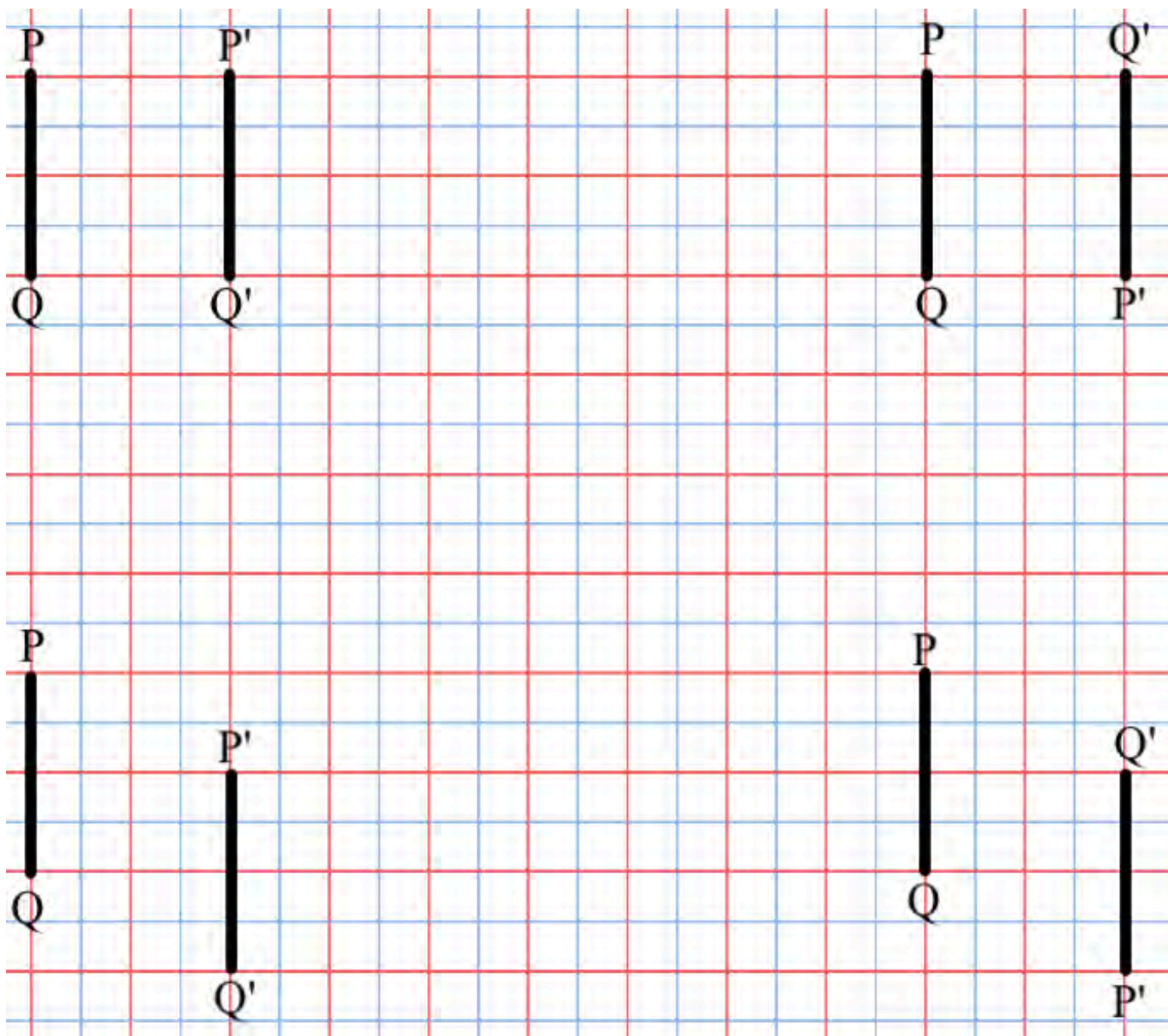
What is the composite of...

6. Two translations (vectors u and v)
7. Two rotations with the same center (angles α and θ)
8. Two dilations with the same center (scaling factors k and r)
9. Two 180° rotations with different centers A and B
10. Two rotations with different centers A and B (angles α and θ)
11. Two dilations with different centers A and B (scaling factors k and r)

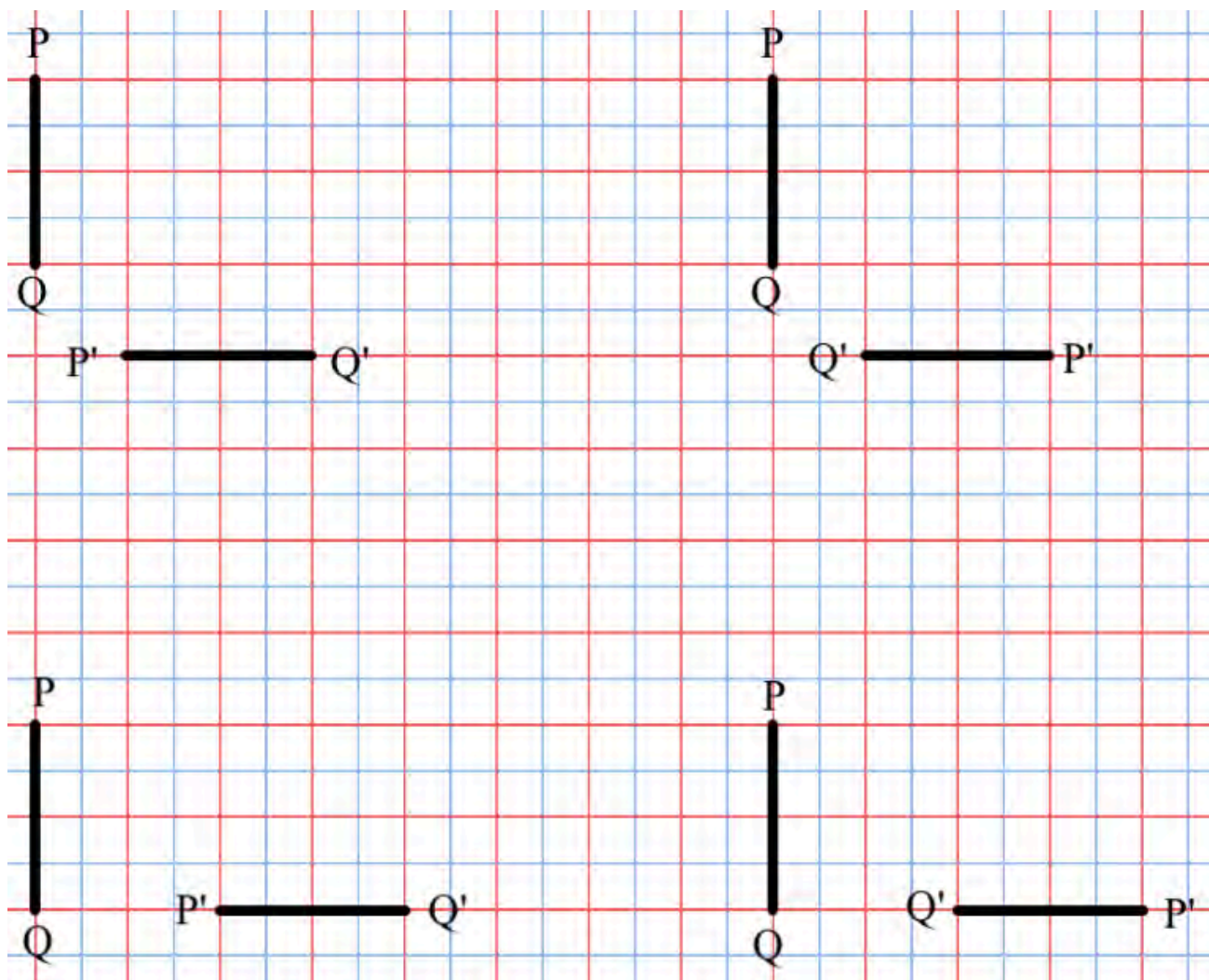
How Many Points?

How many points, and their images, does it take to determine an isometry?

1. Given a point P and its image P' , which basic isometries map P to P' ? Make a list, with as much detail as you can about the isometries. For example, in the case of a rotation, where is its center?
2. Here are some examples involving two points and their images. For each one, find two different basic isometries that map PQ to $P'Q'$. (P must be mapped to P' , Q to Q' . The answers must be a single basic isometry: translation, rotation, reflection, or glide reflection.)



3. Here are more examples involving two points and their images. For each one, find two different isometries that map PQ to $P'Q'$.



These examples show that two points and their images are not enough to identify a unique isometry.

4. In each case, if you had a third point R (not on the line PQ), where might its image R' be? **Hint:** use the fact that distance is preserved.

In #2 and 3, $P'Q'$ was parallel or perpendicular to PQ . In the next lesson we will see that the same situation prevails in the general case.

Three Points

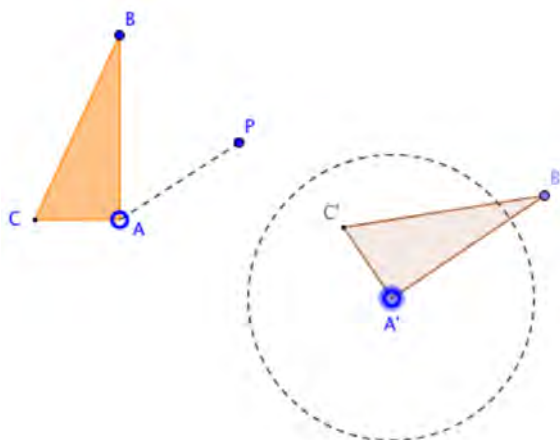
In this lesson we will show the following result:

Theorem: Three non-collinear points and their images determine a unique isometry.

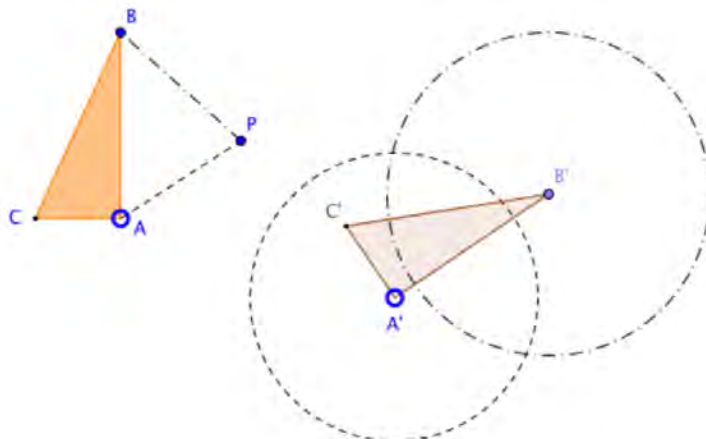
Given: three non-collinear points (A, B, C) and their images in an isometry (A', B', C') .

Prove: The image P' of any point P is uniquely determined.

Step 1: Since isometries preserve distance, $A'P' = AP$. That means that P' must be on a circle centered at A' , with radius AP :



Step 2: By the same logic, since $B'P' = BP$, P' must be on a circle centered at B' , with radius BP .



Therefore P' is on both circles, which means it is in one of two possible locations.

Step 3: To choose the right location, we use the fact that $C'P' = CP$. This implies that P' must be on a third circle, which would help us choose the one correct image. Another way to settle the matter is as follows. If P on the same side of AB as C , then P' is on the same side of $A'B'$ as C' . If P is on the opposite side, then P' is on the opposite side. (If P is on AB , then the two circles are tangent to each other, and we need not choose between two possibilities.)

Note that Step 3 would have been impossible if C was on the line AB . This is why the three given points must not be collinear.

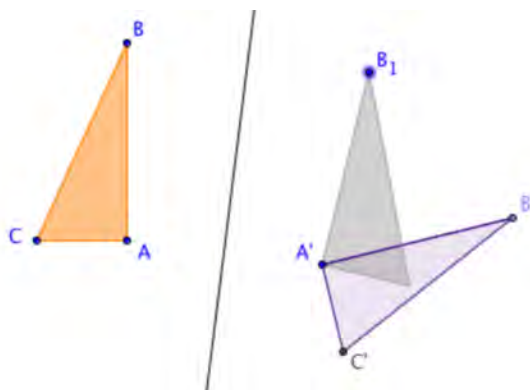
Three Reflections Suffice

Theorem: Any isometry is the composite of at most three reflections.

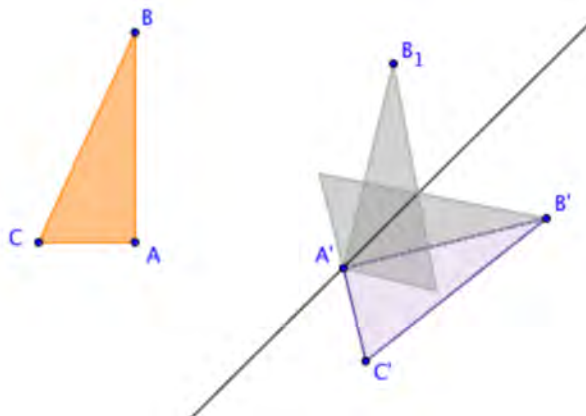
We cannot give a rigorous proof of this theorem yet, but here is a somewhat convincing argument.

Given an isometry M , and a triangle $\triangle ABC$. Say the image of the triangle under M is $\triangle A'B'C'$. Since an isometry is fully determined by three non-collinear points and their images, all we need to prove is that we can get from $\triangle ABC$ to $\triangle A'B'C'$ in three or fewer reflections.

Reflect $\triangle ABC$ across the perpendicular bisector of AA' :



Let B_1 be the image of B . If the triangle's image coincides with $\triangle A'B'C'$, we're done. If not, reflect it across the perpendicular bisector of $B'B_1$. Since A' is equidistant from B and B_1 , it must be on that line, and thus it is not moved by the reflection. By construction, the image of B_1 must be B' .



If the triangle's image coincides with $\triangle A'B'C'$, we're done. If not, reflect it across $A'B'$. This time the image will coincide with $\triangle A'B'C'$.

(A rigorous proof of this theorem is straightforward if we have proved the SSS criterion for congruent triangles.)

One, Two, or Three Reflections

Case 0: If an isometry consists of a single reflection, it is a reflection.

If an isometry is the composition of two reflections, the reflection lines are parallel, or they intersect.

Case 1: two reflections, parallel reflection lines b and c

1. What kind of isometry is it? Be as specific as possible, describing it in terms of the two given lines.

Case 2: two reflections, intersecting reflection lines b and c

2. What kind of isometry is it? Be as specific as possible, describing it in terms of the two given lines.

That leaves the situation where there are three reflection lines. However this situation breaks into multiple cases. For example, all three lines could be parallel.

3. Describe and sketch all the possible cases for three reflections. (**Hint:** there are a total of four cases for the configuration of the lines. One of those breaks down further into three cases, depending on the order that the reflections are performed.)

Only Four Types of Isometries

A detailed analysis of the six cases from #3 shows that:

- ◇ The resulting isometry is a reflection if the lines are all parallel, or all pass through one point. You can find the line of reflection based on the original three lines.
- ◇ In all other cases, the isometry is a glide reflection. You can find the vector and the line of reflection based on the original three lines, but it is a more challenging problem.

The conclusion of such a detailed analysis, combined with the answers about cases 0 through 2 is:

Theorem: Every isometry of the plane is a translation, a rotation, a reflection, or a glide reflection.
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In other words, you can superpose a figure on any congruent figure by using *just one* of these four isometries: translation, rotation, reflection, or glide reflection. No other isometries are needed.

We will complete the proof of this result in the next lesson.

Three Reflections: Six Cases

In this lesson, we complete the proof that any isometry is one of these four: translation, rotation, reflection, or glide reflection. This result follows from the fact that any isometry is the composite of no more than three reflections. We have already analyzed the situation with 1 or 2 reflections. We will now work through the cases with 3 reflections.

1. Show that the composite of reflections in three parallel lines a , b , and c is a reflection. Describe the line of reflection with respect to a , b , and c .
2. Show that the composite of reflections in three concurrent lines a , b , and c is a reflection. Describe the line of reflection with respect to a , b , and c and the point of their intersection.
3. **Challenge:** Assume that exactly two lines among a , b , and c are parallel. Show that the composite of reflections in lines a , b , and c (in that order) is a glide reflection. Describe the line of reflection and the translation vector with respect to a , b , and c .
 - a. If b and c are parallel.
 - b. If a and b are parallel.
 - c. If a and c are parallel.
4. **Challenge:** Assume that lines a , b , and c form a triangle. (They are not concurrent, and no two are parallel.) Show that the composite of reflections in lines a , b , and c is a glide reflection. Describe the line of reflection and the translation vector with respect to a , b , and c .

This completes our proof! Every isometry of the plane is indeed a translation, a rotation, a reflection, or a glide reflection.