Transformational Proof Basics - Similarity

Note: Some of this repeats *Transformational Proof Basics – Congruence*. Repeats are marked with an asterisk (*)

Our proofs are based on the definitions, assumptions, and basic theorems listed here.

Basic Definitions

*Transformation of the Plane: A one-to-one function whose domain and range are the entire plane.
*Isometry (or rigid motion): A transformation of the plane that preserves distance and angles.
Dilation: A dilation with center O and scale factor $k \neq 0$ maps O to itself and any other point P to P' so that O, P, and P' are collinear and the directed segment $OP' = k \cdot OP$. (If k is negative, rays OP' and OP point in opposite directions.
Similarity: Two figures are *similar* if one can be superposed on the other by a sequence of isometries followed by a dilation.
Trapezoid: A quadrilateral with *at least* one pair of parallel sides.
Midsegment of a Triangle: A line segment that connects the midpoints of two of its sides.
Midsegment of a Trapezoid that is not a Parallelogram: A line segment that connects the midpoints of the two non-parallel sides.

Assumptions

The following five assumptions are sufficient for the mathematically experienced. When working with students or developing curriculum, many of the basic theorems proved below can be added to the set of assumptions because many students will think they are obvious.

1. *The parallel postulate: Through a point outside a given line, one and only one line can be drawn parallel to the given line.
2. *Two distinct lines meet in at most one point.
3. *A circle and a line meet in at most two points.
4. *Two distinct circles meet in at most two points.
5. *Reflection preserves distance and angle measure.
6. Dilation preserves collinearity

Similarity Results proved in *Triangle Congruence and Similarity: A Common-Core-Compatible Approach*, by Henri Picciotto and Lew Douglas

(http://www.mathedpage.org/transformations/triangle-congruence-similarity-v2.pdf)

1. Fundamental Theorem of Dilations (FTD): If C, A, and B are not collinear, the image $A'B'$ of the segment AB under a dilation with center C and scale factor k is parallel to AB, with length $k \cdot AB$.
   Corollary**:** Similar triangles have congruent angles and proportional sides.

2. Similarity Criteria for Triangles: SSS, SAS, and AA.

Theorems Proved using Dilation

1. If O is the center of a dilation, k is the scale factor, and O, A, and B are distinct collinear points, then $A'B' = |k|AB$.
   Proof:
   Case 1: A and B are on the same side of O.
   By the definition of dilation and the given information, O, A, A', B, and B' are all collinear. Even if k is negative, A', and B' are on the same side of O. Suppose, without loss of generality, that A is between O and B, so that $OA + AB = OB$. Hence $OA < OB$ and, even if k is negative, $OA' = |k|OA < |k|OB = OB'$. Therefore, A' is between O and B' and $OA' + A'B' = OB'$. By the definition of dilation, $OB' = |k|OB$ and $OA' = |k|OA$. Therefore, $A'B' = OB' - OA' = |k|OB - |k|OA = |k|(OB - OA) = |k|AB$.
   Case 2: A and B are on opposite sides of O.
   The argument is very similar, except that one starts with $AO + OB = AB$. It's still the case that O, A, A', B, and B' are all collinear, but now O is between A' and B'.
   Case 3: Either A or B is the same as O.
   In this case, $A'B' = |k|AB$ by the definition of dilation.

2. Under a dilation, $A'B' = |k|AB$.
   Proof: This was just proved if O, A, and B are collinear. If O, A and B are not collinear, this is the FTD.

3. Dilation preserves betweenness.
   Proof: If points A, B and C are in order on a line, $AB + BC = AC$. If k is the scale factor of the dilation, then $A'B' = |k|AB$, $B'C' = |k|BC$, and $A'C' = |k|AC$. Therefore, $A'B' + B'C' = A'C'$. This implies that B' is between A' and C'.

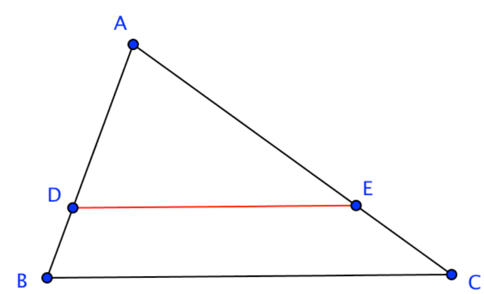
4. The image of a segment under a dilation is a segment, the image of a ray is a ray, and the image of a line is a line.
   Proof: This is true because collinearity and betweenness are preserved, and because dilation is one-to-one.

5. Dilation preserves angle measure.
   Proof: The image of any ray under a dilation is another ray that is either parallel to, or collinear with, its pre-image. The same is true for translation, so one can translate any angle into its image under a dilation. Since translations preserve angles, dilations do too.

6. Dilation preserves the ratio of the lengths of any two segments.
   Proof: Let AB and CD be two segments. $A'B' = |k|AB$ and $C'D' = |k|CD$ implies that $\frac{A'B'}{C'D'} = \frac{AB}{CD}$.

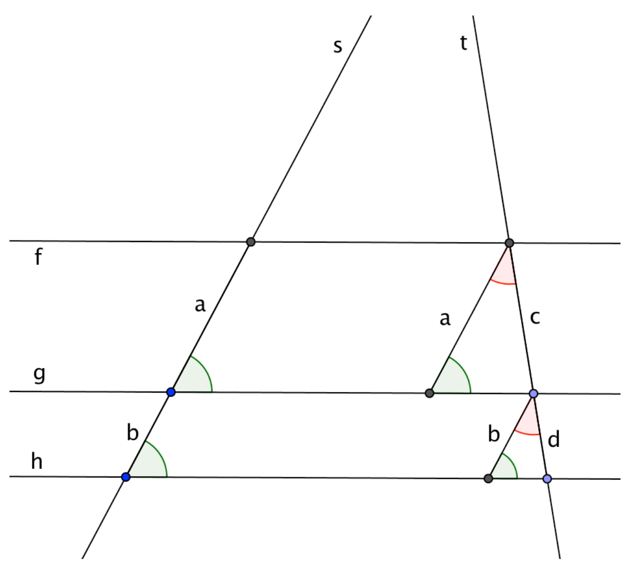
7. The midsegment of a triangle is parallel to the third side and half as long.
Proof: Let $\overline{DE}$ be a midsegment of ΔABC. Dilate ΔABC from A with scale factor $k = \frac{1}{2}$. Since D and E are midpoints, by the definition of dilation, B' = D and C' = E. The FTD tells us that $\overline{DE}||\overline{BC}$ and $DE = \frac{1}{2}BC$.

8. If a segment joins points on two sides of a triangle whose distances are the same fraction k ($0 < k < 1$) from their common endpoint to their other endpoint, then the segment joining these points is parallel to the third side and its length is the same fraction k of it.
Proof: Let D and E be points on sides AB and AE of ΔABC respectively, chosen so that $AD = kAB$ and $AE = kAC$, where $0 < k < 1$.
Dilate ΔABC from A with scale factor k. By the definition of dilation, B' = D and C' = E. The FTD tells us that DE || BC and $DE = kBC$.
Note: A similar argument works if $k > 1$ if you replace segments AB and AC by rays AB and AC.

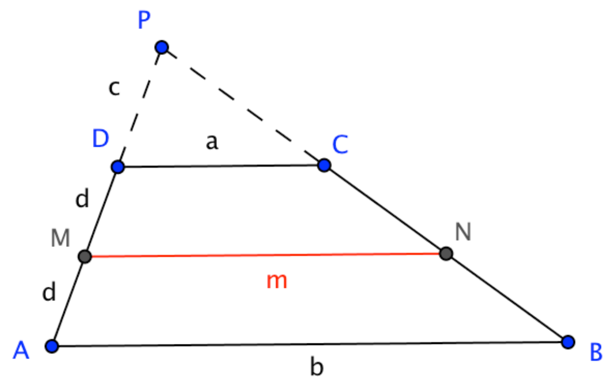


9. If three parallel lines cut off segments whose lengths have ratio r on one transversal, they cut off segments whose lengths have the same ratio r on any transversal.
Note: The proof of Theorem 9 involves similar triangles, not dilation. It is included because Theorem 9 is a lemma for Theorem 10.
Proof: The parallel lines are f, g, and h; the transversals are s and t. Two segments are drawn parallel to s as shown. a's and b's are marked to show that the opposite sides of the parallelograms are equal. By properties of angles formed by parallel lines, all the green angles are equal, and the two red angles are equal. Since the triangles are similar by AA, $\frac{a}{b} = \frac{c}{d}$.

Corollary: If three parallel lines cut off segments of equal length on one transversal, they cut off segments of equal length on any transversal.



10. If a trapezoid is not a parallelogram, its midsegment is parallel to the bases and its length is their average.



Proof: In trapezoid *ABCD*, let *M* be the midpoint of leg *AD*. Since *AD* is not parallel to *BC*, we can extend the legs to meet at *P*. Dilate *AB* from P with scale factor $\frac{c+d}{c+2d}$. By the FTD and the definition of dilation, *N* lies on *PB* and *MN* || *AB*. *AM* = *MD*, so by Theorem 9, *BN* = *NC* and *N* is the midpoint of side *BC*. Therefore, $\overline{MN}$ is a midsegment parallel to the bases of trapezoid *ABCD*. It follows that $\frac{c}{a} = \frac{c+d}{m} = \frac{c+2d}{b}$.

Now we use an algebraic result that is easy to verify: If $\frac{x}{y} = \frac{v}{w}$, then $\frac{x}{y} = \frac{v}{w} = \frac{v-x}{w-y}$. In words, if you have two equivalent fractions, subtracting (or adding as it turns out) numerators and denominators yields another equivalent fraction. Applying this to the first two and last two fractions in $\frac{c}{a} = \frac{c+d}{m} = \frac{c+2d}{b}$ gives $\frac{d}{m-a} = \frac{d}{b-m}$. Equating denominators and solving for *m* gives $m = \frac{a+b}{2}$.

Note: If the trapezoid IS a parallelogram, then the segment joining midpoints of any opposite sides has the same length as those sides, so is again the average. The result is true for any trapezoid, even with our inclusive definition.