Symmetry Definitions and Properties - Triangles and Quadrilaterals

This document is a reference for teachers and curriculum developers. It is based on a choice to rethink the geometry curriculum on a transformational foundation. It is a sequel to our "Triangle Congruence and Similarity: A Common-Core-Compatible Approach" (available on http://www.mathedpage.org/transformations/). As you will see, this approach implies some changes in the hierarchy of quadrilaterals.

We recommend that you read *Transformation Proof Basics* first. It contains the definitions, assumptions, and lemmas (simple, helping theorems) on which these proofs are based.

1. Isosceles Triangle: A triangle with one line of symmetry.

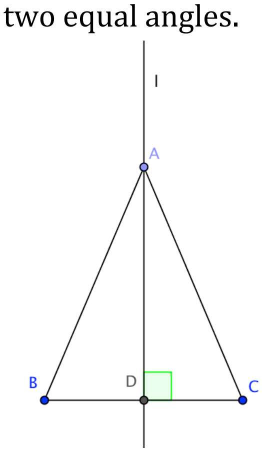
The etymology of "isosceles", of course, is "equal legs". In the scheme we propose, this is no longer the definition: it must be proved. (See Property c below.)

Properties:

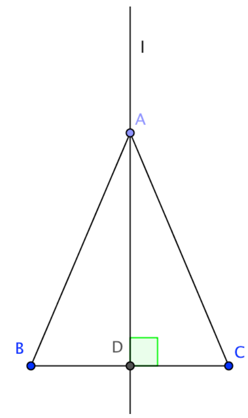
a. The image of a vertex in a line of symmetry is also a vertex.
Proof: A vertex is the common endpoint of two sides. Because collinearity is preserved, sides must map onto sides. So, the image of a vertex must also lie on two sides. A point on two sides is a vertex, so it must also be a vertex.

b. One vertex lies on the line of symmetry and the other two are each other's reflections.
Proof: Because there is an odd number of vertices, one of them must lie on the line of symmetry.

c. An isosceles triangle has two equal sides and two equal angles.
Proof: Reflection preserves side lengths and angle measure. If vertex A is on the line of symmetry, then AB = AC and ∠B = ∠C.

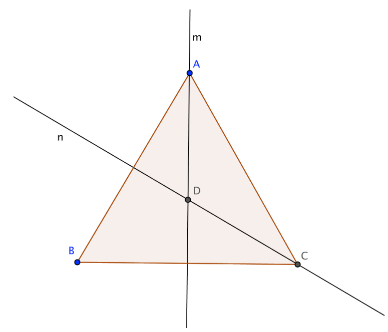


d. The perpendicular bisector of the third side of an isosceles triangle bisects an angle of the triangle, so the line of symmetry is an altitude, a median, and a perpendicular bisector.



Proof: By the definition of reflection, the line of symmetry *l* is the perpendicular bisector of BC. It also must pass through A. Since reflection preserves angles, ∠DAB = ∠DAC, so *l* bisects ∠BAC. *l* is clearly an altitude, median, and perpendicular bisector.

2. Equilateral Triangle: A triangle with two lines of symmetry.



Other (equivalent) definitions are possible. We prefer this one, as it is economical, and facilitates the proof of properties.

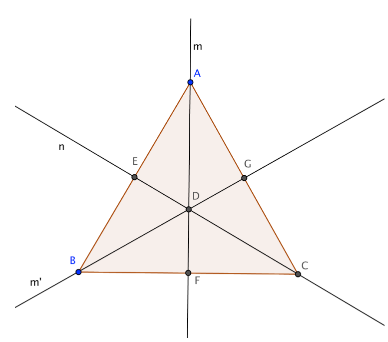
Note that once again, the etymology does not correspond to the definition; we must prove all sides are equal. (See property b below.)

Properties:

a. An equilateral triangle has 3-fold rotational symmetry.
Proof: Let *m* and *n* be the symmetry lines through A and C respectively. The composition of reflections in *m*, then *n* maps the triangle onto itself and is a rotation around their intersection point D. Call this rotation *r*. *r* maps A onto B, B onto C, and C onto A. Repeating this rotation three times gives the identity transformation, so the triangle has 3-fold rotational symmetry around the intersection point of its two lines of symmetry.

b. All sides of an equilateral triangle are equal and each angle is 60˚.
Proof: Rotation preserves side lengths and angle measure. Since the sum of the angles in a triangle is 180˚, each angle is 60˚.

c. An equilateral triangle has three concurrent lines of symmetry.
Proof: *r* maps A to B and D to itself, so *m'*, the image of *m* under *r*, passes through B and D. Since *m* perpendicularly bisects BC, *m'* must perpendicularly bisect CA because rotation preserves segment length and angle measure. Therefore *m'* is a third line of symmetry of ΔABC.



d. Each line of symmetry of an equilateral triangle is an altitude, a median, and a perpendicular bisector.
Proof: The triangle is isosceles in three different ways.

3. Parallelogram: A quadrilateral with 2-fold rotational symmetry.
Note: This figure and a general trapezoid are the only special quadrilaterals whose definitions do not involve line symmetry.

Properties:

a. The image of a vertex under the symmetry rotation is an opposite vertex.
Proof: Let *r* be the 2-fold rotation. *r* followed by *r* ($r \circ r$) is a 360° rotation, i.e. the identity. As with triangles, the image of a vertex under *r* must be a vertex. Its image under $r \circ r$ must be itself. If the image were a consecutive vertex, then the image under $r \circ r$ would be the next consecutive vertex (i.e. the opposite vertex), not the original. Therefore, the image is the opposite vertex.
(This argument is subtle, and the result is obvious enough that we recommend not including it in discussions with students. They will be willing to accept this result without a proof, and would most likely find the proof more confusing than illuminating. The same is true of property (d) below.)

b. The center of the 2-fold rotation is the common midpoint of the diagonals.
Proof: A diagonal must rotate into itself because its endpoints switch. So, a diagonal must contain the center of rotation. The distance from the center to one endpoint must equal the distance to the other because rotation preserves distance. Therefore, the center must be the midpoint of either diagonal, which implies it is the common midpoint of both.

c. The opposite sides of a parallelogram are parallel.
Proof: The image of a line under a half-turn around a point not on the line is a parallel line.

d. Consecutive angles of a parallelogram are supplementary
Proof: This is a property of parallel lines cut by a transversal.

e. The image of a side under *r* is an opposite side. The image of an angle under *r* is an opposite angle.
Proof: The image can't be a consecutive side because then it wouldn't be parallel to the pre-image. The image of an angle can't be a consecutive angle because then one image side wouldn't be parallel to

its pre-image.

f. The opposite sides and opposite angles of a parallelogram are equal.
Proof: Rotation preserves segment length and angle measure.

4. General Trapezoid: A quadrilateral where one side is a dilation or translation of the opposite side. In the translation case, the trapezoid is a parallelogram because one pair of opposite sides is both parallel and equal. (This will be proved later.)

Properties:

a. A pair of opposite sides (called *bases* in the dilation case) are parallel.
Proof: If a dilation, the center of dilation can't be on a line containing a side, because if it were, the opposite sides would be collinear. By the FTD, the image of a side is parallel to its pre-image. If a translation, the image of a line under translation by a vector not parallel to the line is a line parallel to its pre-image.
b. Consecutive angles (on different bases if this is not a parallelogram) are supplementary.
Proof: This is a property of parallel lines cut by a transversal.
c. All pairs of consecutive angles of a parallelogram are supplementary (Theorem 3d above).

5. Isosceles Trapezoid: A quadrilateral with a line of symmetry though interior points of opposite sides. These sides are called *bases*. The other two sides are called *legs*.

Properties:

a. Two vertices of an isosceles trapezoid are on one side of the symmetry line and two are on the other.
Proof: Since reflection maps vertices to vertices, the four vertices must be evenly split on both sides of the symmetry line.

b. The symmetry line of an isosceles trapezoid is the perpendicular bisector of bases.
Proof: One endpoint of each of these sides must reflect into the other. A reflection line perpendicularly bisects the segment joining pre-image and image points if these points are not on the reflection line.

c. The bases of an isosceles trapezoid are parallel.
Proof: They are both perpendicular to the symmetry line. Two distinct lines perpendicular to the same line are parallel.

d. The legs of an isosceles trapezoid are equal.
Proof: Reflection preserves segment length.

e. Two consecutive angles of an isosceles triangle on the same base are equal.
Proof: Reflection preserves angle measure.

f. The diagonals of an isosceles trapezoid are equal.
Proof: One diagonal reflects to the other. Reflection preserves segment length.

g. The intersection point of the equal diagonals of an isosceles trapezoid lies on the symmetry line.
Proof: The point where one diagonal intersects the symmetry line must be invariant under reflection in the symmetry line because it lies on it. Therefore, it also lies on the other diagonal.

h. The intersection point of the diagonals of an isosceles trapezoid divides each diagonal into equal subsections.
Proof: The subsections of one diagonal determined by the intersection point reflect onto the subsections of the other. These subsections are equal because reflection preserves segment length.

6. Kite: A quadrilateral with a line of symmetry through opposite vertices. (It is possible to omit "opposite" from the definition and prove that if a line of symmetry passes through vertices, they must be opposite. But for most students, this sort of subtlety would be counterproductive. On the other hand, it could be an optional challenge.)

   Properties:

   a. A kite has two disjoint pairs of consecutive equal sides and one pair of equal opposite angles. (We need to say "disjoint," because the pairs can't have a common side.)
   Proof: Reflection in the line of symmetry preserves segment length and angle measure.

   b. The line of symmetry of a kite bisects a pair of opposite angles.
   Proof: Reflection preserves angle measure.

   c. The diagonal of a kite that lies on the line of symmetry perpendicularly bisects the other diagonal.
   Proof: The symmetry line perpendicularly bisects the segment joining the vertices not on the line because they reflect into each other.

7. Rhombus: A quadrilateral with two lines of symmetry passing through opposite vertices. (So, a rhombus is a kite in two different ways.)

   Properties:

a. A rhombus has all sides equal and two pairs of equal opposite angles.
Proof: A kite has two disjoint pairs of consecutive equal sides and one pair of equal opposite angles. The result follows because a rhombus is a kite in two different ways (i.e. both diagonals are lines of symmetry).

b. Each diagonal of a rhombus bisects its angles.
Proof: Each line of symmetry bisects a pair of opposite angles (property of kites).

c. The diagonals of a rhombus perpendicularly bisect each other.
Proof: The diagonal of a kite that lies on the line of symmetry perpendicularly bisects the other diagonal. For a rhombus, each diagonal has this property.

d. A rhombus is a special parallelogram.
Proof: Since a rhombus has two perpendicular lines of symmetry, the composition of reflection in these lines is a 180˚ rotation around their point of intersection. (The composition of two reflections is a rotation around their point of intersection through twice the angle between the reflection lines.) Since each reflection maps the rhombus onto itself, their composition does also.

e. The opposite sides of a rhombus are parallel.
Proof: Since a rhombus is a parallelogram, the opposite sides are parallel.

8. Rectangle: A quadrilateral with two lines of symmetry passing through interior points of the opposite sides. (So, a rectangle is an isosceles trapezoid in two different ways.)

Properties:

a. The symmetry lines of a rectangle perpendicularly bisect the opposite sides.
Proof: A rectangle is an isosceles trapezoid in two different ways.

b. A rectangle is equiangular.
Proof: Two consecutive angles of an isosceles trapezoid that share a base are equal. Both pairs of opposite sides are bases because of the two different ways, so any two consecutive angles share a base.

c. All angles of a rectangle are right angles.
Proof: The sum of the interior angles of any quadrilateral is 360˚ and $360 \div 4 = 90$.

d. The symmetry lines of a rectangle are perpendicular.
Proof: The lines divide the rectangle into four quadrilaterals. Each has

three right angles: one is an angle of the rectangle and the other two are formed by a side and a symmetry line, which are perpendicular. Since the sum of the angles of a quadrilateral is 360˚, the fourth angle at the intersection of the symmetry lines must also be a right angle.

e. A rectangle has 2-fold rotational symmetry, so it is a special parallelogram.
Proof: Reflecting a rectangle in one line of symmetry followed by the other maps the rectangle onto itself and is equivalent to a 180˚ rotation because the symmetry lines meet at right angles. Therefore, a rectangle has 2-fold rotational symmetry around the intersection of the symmetry lines.

f. The opposite sides of a rectangle are parallel and equal.
Proof: These are properties of a parallelogram. A rectangle is a special parallelogram.

g. The diagonals of a rectangle are equal.
Proof: This is a property of an isosceles trapezoid. A rectangle is a special isosceles trapezoid.

h. The diagonals of a rectangle bisect each other.
Proof: This is a property of a parallelogram. A rectangle is a special parallelogram.

i. The diagonals of a rectangle and the lines of symmetry are all concurrent.
Proof: The intersection point of the equal diagonals of an isosceles trapezoid lies on the symmetry line. For a rectangle, the intersection point lies on both symmetry lines, so it is their intersection.

9. Square: A quadrilateral with four lines of symmetry: two diagonals and two lines passing through interior points of opposite sides.

Properties:

a. A square is a special rectangle, rhombus, kite, and isosceles trapezoid, so it inherits all the properties of these quadrilaterals.
Proof: True, by definition of a square.

b. If a square and all four symmetry lines are drawn, all the acute angles are 45˚.
Proof: The diagonals bisect the interior right angles because a square is a rhombus. All eight right triangles formed have a right angle where the symmetry lines intersect the sides and a 45˚ angle where they intersect the vertices. Since the sum of the angles of a triangle is 180˚,

the remaining angles at the center must all be 45˚.

c. A symmetry line through sides and a symmetry line through vertices form a 45˚ angle.
Proof: An immediate consequence of the result just above.

d. A square has 4-fold rotational symmetry.
Proof: Reflecting a square in a line of symmetry through the sides followed by a line of symmetry through the vertices maps the square onto itself. It is equivalent to a 90˚ rotation because these symmetry lines meet at a 45˚angle. Therefore, a square has 4-fold rotational symmetry around its center (the intersection point of the lines of symmetry).