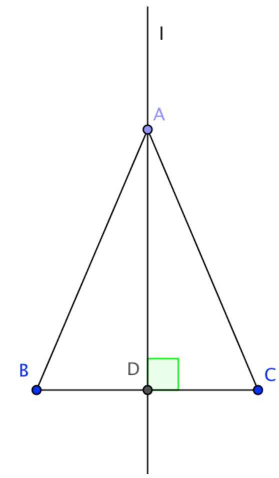
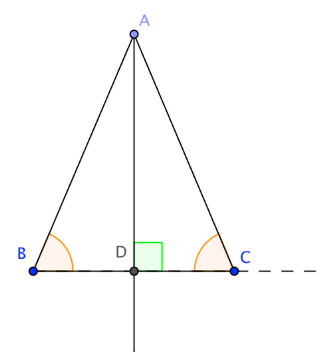
Proving Triangles and Quadrilaterals Satisfy Transformational Definitions

1. Definition of Isosceles Triangle: A triangle with one line of symmetry.

a. If a triangle has two equal sides, it is isosceles.
Proof: Let AB and AC be the equal sides. A must lie on the perpendicular bisector l of BC because it is equidistant from B and C. By the definition of reflection, $B' = C$ under reflection in l. $A' =$ A because it lies on l. Therefore, l is a line of symmetry for ΔABC.

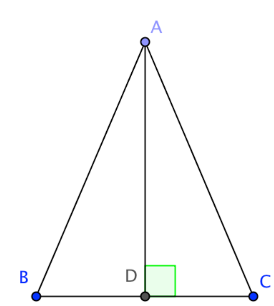


b. If a triangle has two equal angles, it is isosceles.
Proof: Draw ray AD, the angle bisector of $\angle BAC$. Two angles of ΔBAD and ΔCAD are equal, so the third angles ($\angle BDA$ and $\angle CDA$) must be equal. Since they are supplementary, they are both right angles. Reflect B in ray AD. Since reflections preserve angles, B' must be on ray AC. By the definition of reflection, B' must be on ray BD. Therefore, B' is at the intersection of ray BD and ray AC, which is C. Therefore, $B' = C$. Since C is the reflection of B in line AD, and A is its own reflection in line AD, AD is a line of symmetry for the triangle.



c. If an angle bisector of a triangle is also an altitude, the triangle is isosceles.
Proof: Let l, the bisector of $\angle BAC$ be perpendicular to side BC at D, so that $\angle DAB = \angle DAC$ and $\angle ADB = \angle ADC$. Two angles of ΔBAD and ΔCAD are equal, so the third angles ($\angle B$ and $\angle C$) must be equal. By Theorem 1b above, the triangle is isosceles.

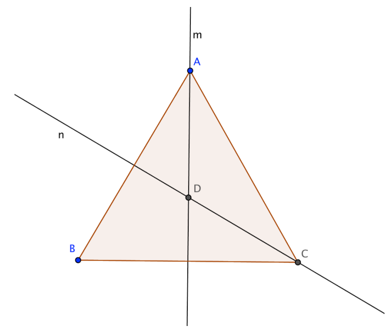
d. If an altitude of a triangle is also a median, the triangle is isosceles.
Proof: Since altitude AD is also a median, AD is the perpendicular bisector of BC. Since any point on the perpendicular bisector of a segment is equidistant from the endpoints, $AB = AC$. By Theorem 1a above, the triangle is isosceles.



e. If an angle bisector of a triangle is also a median, the triangle is isosceles.
<u>Proof</u>: Postponed until the end of the rhombus section (after Theorem 6e), because a rhombus is constructed in the proof.

2. <u>Definition of Equilateral Triangle</u>: A triangle with two lines of symmetry.

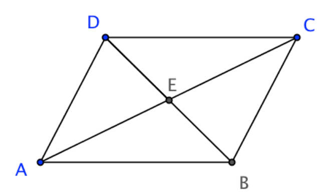
a. If a triangle has all *sides* equal, then it's an equilateral triangle.
<u>Proof</u>: In ΔABC, $AB = BC = CA$. Since $AB = AC$, the triangle is isosceles with symmetry line m. Since $CA = CB$, the triangle is isosceles with symmetry line n. Since it has two lines of symmetry, it is equilateral.



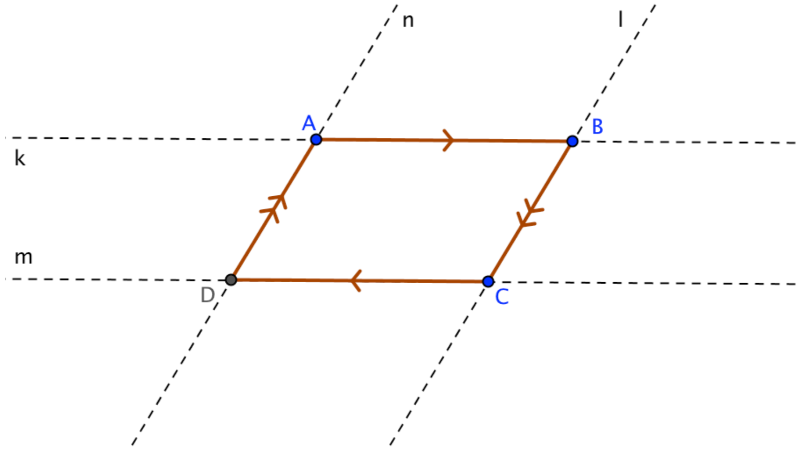
b. If a triangle has all *angles* equal, then it's an equilateral triangle.
<u>Proof</u>: The argument is virtually identical to the previous one, but uses Theorem 1b instead of 1a.

3. <u>Definition of Parallelogram:</u> A quadrilateral with 2-fold rotational symmetry.

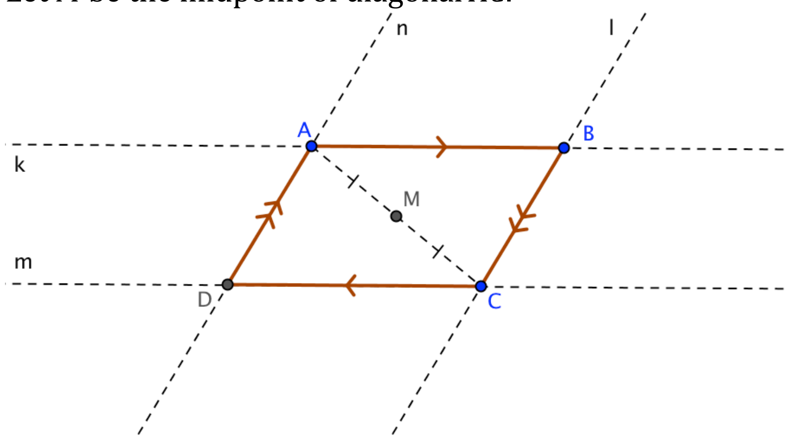
a. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
<u>Proof</u>: Rotate quadrilateral $ABCD$ 180˚ around point E, the intersection of the diagonals. Since the rotation is 180˚, B' lies on ray ED. Since rotation preserves distance, $B' = D$. Similarly, $A' = C$. Similarly, $C' = A$ and $D' = B$. Because rotation maps segments to segments, each side of $ABCD$ maps to the opposite side. Therefore, quadrilateral ABCD has 2-fold rotational symmetry. By definition, $ABCD$ is a parallelogram.



b. If opposite sides of a quadrilateral are parallel, the quadrilateral is a parallelogram.
<u>Proof</u>: Given quadrilateral *ABCD* as in this figure, with sides extended to lines *k*, *l*, *m*, and *n*. We would like to prove that it has 2-fold (half-turn) symmetry.

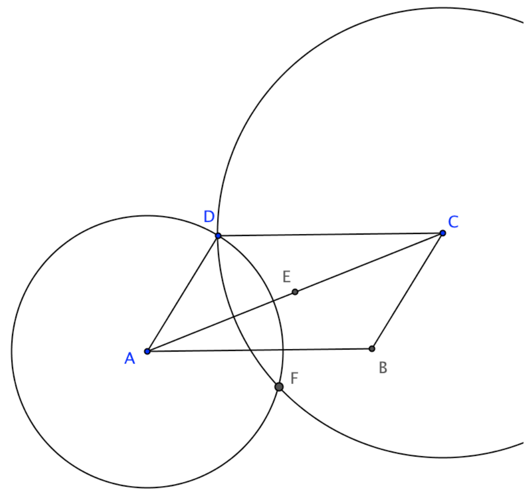


Let *M* be the midpoint of diagonal *AC*.

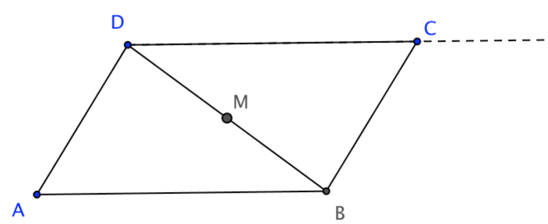


Consider *H*, the half-turn with center *M*. Since *M* is the midpoint of segment *AC*, *A'* = *C* and *C'* = *A* under *H*. Because *M* is on neither *k* nor *l*, their image lines are parallel to their pre-images. Because of the parallel postulate, there is only one parallel to *k* through *C* and one parallel to *l* through *A*. Therefore, *k'* = *m*, and *l'* = *n*. *B* is the intersection of lines *k* and *l*, and therefore its image is the intersection of lines *m* and *n*, which is *D*. Since *A'* = *C* and *B'* = *D*, *ABCD* has 2-fold rotational symmetry. By definition, *ABCD* is a parallelogram.

c. If opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.
<u>Proof</u>: In quadrilateral *ABCD*, draw diagonal *AC* and its midpoint *E*. Under a half turn around *E*, *A'* = *C* and *C'* = *A*. Since *CB* = *AD*, B' lies on circle *A* with radius *AD*. Since *AB* = CD, *B'* lies on circle *C* with radius *CD*. These circles intersect at *D* and *F*. But *F* is on the same side of line *AC* as *B*, so *B'* ≠ *F*. Therefore *B'* = *D* and *ABCD* has 2-fold symmetry around *E*. By definition, *ABCD* is a parallelogram.

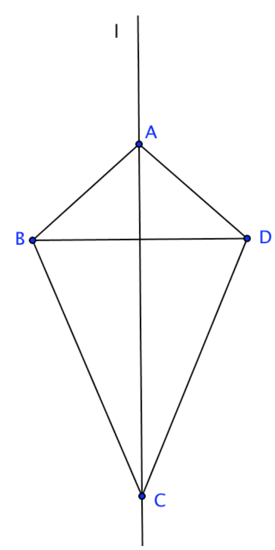


d. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.
<u>Proof</u>: In quadrilateral *ABCD*, suppose *AB* is equal and parallel to *DC*. Draw diagonal *BD* and its midpoint *M*. Under a half-turn around *M*, *B'* = *D* and *D'* = *B*. The image of ray *BA* is parallel to *AB*, so it must coincide with ray *DC*. Because *AB* = *DC*, that means that *A'* = *C*, and therefore *C'* = *A*. Hence *ABCD* is a parallelogram.

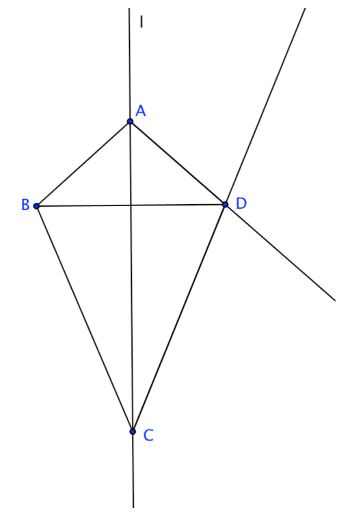


4. <u>Definition of Kite:</u> A quadrilateral with one line of symmetry through opposite vertices.

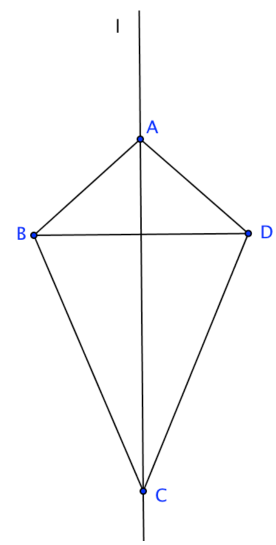
   b. If two disjoint pairs of consecutive sides of a quadrilateral are equal, the quadrilateral is a kite.
   <u>Proof</u>: In quadrilateral *ABCD*, suppose *AB* = *AD* and *CB* = *CD*. Since *A* and *C* are both equidistant from *B* and *D*, they lie in the perpendicular bisector of diagonal *BD*. Therefore, *l* is the perpendicular bisector of diagonal *BD*. Under reflection in *l*, *B'* = *D* and *D'* = *B*. Because *A* and *C* both lie on *l*, *A'* = *A* and *C'* = *C*. *l* is therefore a line of symmetry and *ABCD* is a kite.



b. If a diagonal of a quadrilateral bisects a pair of opposite angles, the quadrilateral is a kite.
Proof: Label as *l* the line through diagonal *AC* that bisects ∠*BAD* and ∠*BCD*. Consider reflection in *l*. Since *A* and *C* are on *l*, *A′* = *A* and *C′* = *C*. Since ∠*BAC* = ∠*DAC*, *B′* lies on ray *AD*. Since ∠*BCA* = ∠*DCA*, *B′* lies on ray *CD*. Because these rays intersect at *D*, *B′* = *D*, which implies that *D′* = *B*. Therefore, *l* is a line of symmetry and *ABCD* is a kite.

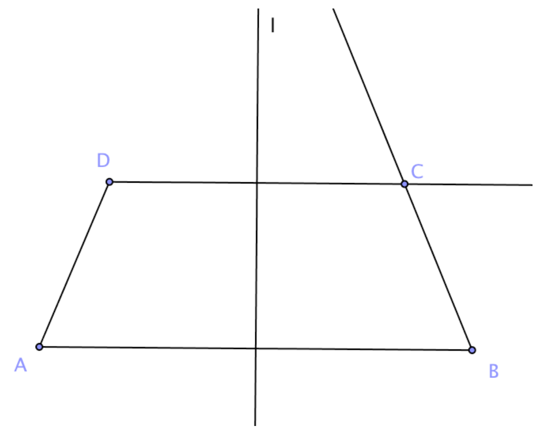


c. If one diagonal of a quadrilateral perpendicularly bisects the other, the quadrilateral is a kite.
Proof: In quadrilateral *ABCD*, diagonal *AC* perpendicularly bisects diagonal *BD*. Let *l* be the line through *A* and *C*, and reflect *ABCD* in *l*. Since *A* and *C* lie on *l*, *A′* = *A* and *C′* = *C*. By the definition of reflection, *B′* = *D* and *D′* = *B*. Therefore, *l* is a line of symmetry and *ABCD* is a kite.

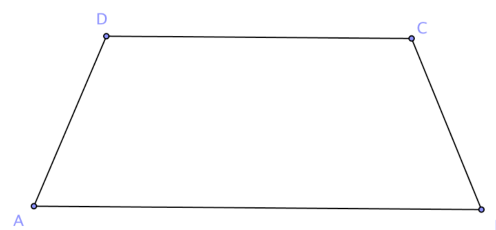


5. Definition of Isosceles Trapezoid: A quadrilateral with a line of symmetry though midpoints of opposite sides.

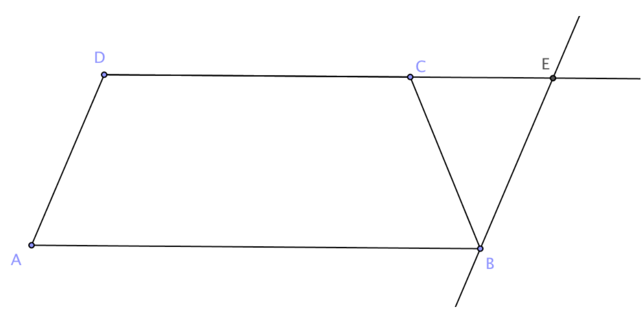
   a. If one pair of opposite sides of a quadrilateral are parallel and a pair of consecutive angles on one of these sides are equal, the quadrilateral is an isosceles trapezoid.
   Proof: In quadrilateral *ABCD*, *DC* || *AB* and ∠*A* = ∠*B*. Let *l* be the perpendicular bisector of *AB*. Under reflection in *l*, *A*′ = *B* and *B*′ = *A*. Since *DC* || *AB*, *DC*⊥*l*. Therefore, *D*′ lies on ray *DC*. Because *A*′ = *B*, ray *AB* maps to ray *BA*, ∠*A* = ∠*B*, and reflection preserves angles, *D*′ lies on ray *BC*. These rays intersect at *C*, so *D*′ = *C*. Thus, *l* is a line of symmetry for *ABCD* and *ABCD* is an isosceles trapezoid.



   b. If two disjoint pairs of consecutive angles of a quadrilateral are equal, the quadrilateral is an isosceles trapezoid.
   Proof: In quadrilateral *ABCD*, ∠*B* = ∠*A* and ∠*C* = ∠*D*. Because ∠*A* + ∠*B* + ∠*C* + ∠*D* = 360˚, 2∠*A* + 2∠*D* = 360˚. Dividing both sides by 2 gives ∠*A* + ∠*D* = 180˚, so *DC* || *AB*. By Theorem 5a, *ABCD* is an isosceles trapezoid.

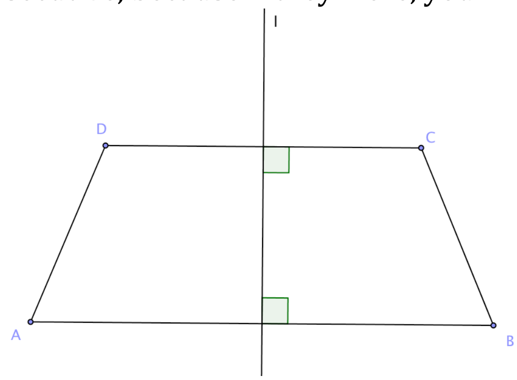


   c. If two opposite sides of a quadrilateral are parallel and if the other two sides are equal but not parallel, then the quadrilateral is an isosceles trapezoid.
   Proof: In quadrilateral *ABCD*, *DC* || *AB*, *AD* = *BC*, and *AD* is not parallel to *BC*. Through *B*, draw a line parallel to *AD* meeting ray *DC* at *E*. Since *ABDE* has two pairs of opposite parallel sides, it is a parallelogram. Because the opposite angles of a parallelogram are equal, ∠A = ∠*BEC*. The opposite sides of a parallelogram are also equal, so *AD* = *BC* = *BE*. If two sides of a triangle are equal, the triangle is isosceles, which implies that ∠*BEC* = ∠*BCE*. Finally, because *DC* || *AB*, ∠*BCE* = ∠*ABC*. The chain of equal angles now reads

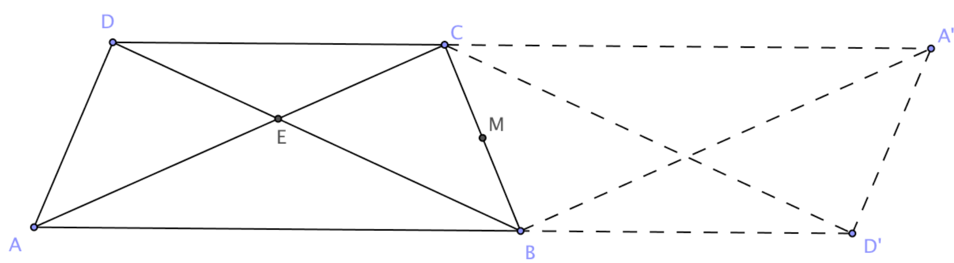


$\angle A = \angle BEC = \angle BCE = \angle ABC$. This means that *ABCD* has a pair of consecutive equal angles on one of its parallel sides. By Theorem 5a, *ABCD* is an isosceles trapezoid.

d. If a line perpendicularly bisects two sides of a quadrilateral, the quadrilateral is an isosceles trapezoid.
<u>Proof</u>: The two sides can't be consecutive, because if they were, you would have two consecutive parallel sides, which is impossible. In quadrilateral *ABCD*, *l* is the perpendicular bisector of *AB* and *DC*. Under reflection in *l*, therefore, $D' = C$ and $A' = B$. Thus, *l* is a line of symmetry and *ABCD* is an isosceles trapezoid.



e. If two sides of a quadrilateral are parallel, and if the diagonals are equal, the quadrilateral is an isosceles trapezoid.
<u>Proof</u>:



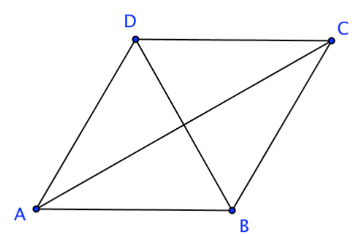
In quadrilateral *ABCD*, $DC \parallel AB$ and $AC = BD$. Let *M* be the midpoint of *BC*. Rotate *ABCD* 180° around *M*. Since $B' = C$ and $C' = B$, $BD' \parallel DC$, and $CA' \parallel AB$, A' is on ray *DC* and D' is on ray *AB*. Because rotation preserves segment length, $BD = CD'$. Therefore, $AC = BD = CD'$. Consider $\Delta ACD'$. Since two sides are equal, it is isosceles, so $\angle BAC = \angle BD'C$. *BD* is also rotated 180° around *M*, so $D'C \parallel BD$. Using transversal AD', we see that $\angle BD'C = \angle ABD$. Thus $\angle BAC = \angle BD'C = \angle ABD$. Because two angles in ΔABE are equal, the triangle is isosceles, which implies that $AE = BE$. In other words, *E* is equidistant from *A* and *B*, so it must lie on the perpendicular bisector of *AB*. A similar argument shows that *E* lies on the perpendicular bisector of *DA*. Since the perpendicular bisectors of *AB* and *DC* pass through the same point *E*, they coincide. This perpendicular bisector is therefore a line of symmetry for *ABCD*, so it is an isosceles trapezoid by definition.

6. <u>Rhombus</u>: A quadrilateral with two lines of symmetry passing through opposite vertices. (So, a rhombus is a kite in two different ways.)

   a. If the diagonals of a quadrilateral perpendicularly bisect each other, the quadrilateral is a rhombus.
   <u>Proof</u>: By the definition of reflection, the two vertices not on either diagonal are images of each other under reflection in that diagonal. Therefore, both diagonals are lines of symmetry, which is the definition of a rhombus.

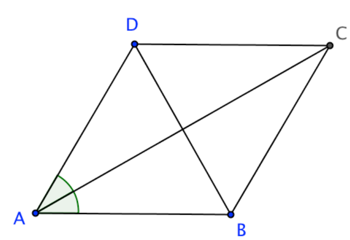
   b. If a quadrilateral is equilateral, it is a rhombus.
   <u>Proof</u>: Since opposite sides are equal, the quadrilateral is a parallelogram. Therefore, the diagonals bisect each other. Since two disjoint pairs of consecutive sides are equal, the quadrilateral is a kite. Therefore, the diagonals are perpendicular. Now we know that the diagonals perpendicularly bisect each other, so the quadrilateral is a rhombus by Theorem 6a.

   c. If both diagonals of a quadrilateral bisect a pair of opposite angles, the quadrilateral is a rhombus.
   <u>Proof</u>: Consider the diagonals separately. Since $\angle ABC$ and $\angle ADC$ are bisected, $ABCD$ is a kite with $AD = CD$ and $AB = CB$. Since $\angle BAD$ and $\angle BCD$ are bisected, $ABCD$ is a kite with $AD = AB$ and $CD = CB$. Therefore, all four sides are equal and the quadrilateral is a rhombus by Theorem 6b.



   d. If both pairs of opposite sides of a quadrilateral are parallel, and if two consecutive sides are equal, the quadrilateral is a rhombus.
   <u>Proof</u>: Since both pairs of opposite sides are parallel, the quadrilateral is a parallelogram. Therefore, both pairs of opposite sides are equal. Since a pair of consecutive sides are equal, all four sides must be equal. Hence the quadrilateral is a rhombus by Theorem 6b.

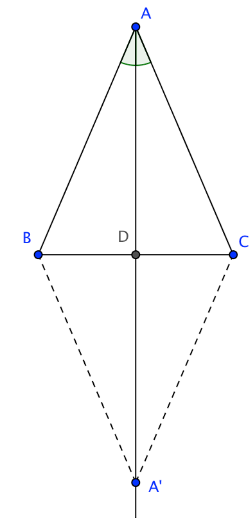
   e. If a diagonal of a parallelogram bisects an angle, the parallelogram is a rhombus.
   <u>Proof</u>: In parallelogram $ABCD$, diagonal AC bisects $\angle DAB$. The opposite sides of a parallelogram are parallel, so this implies that $\angle DCB$ is bisected as well by angle properties of parallel lines. Reflect B across diagonal AC. Because of the equal angles, B' lies on both ray AD and ray CD, so $B' = D$. Since reflection preserves distance, $AD = AB$. But the opposite sides of a



parallelogram are equal, so *ABCD* is equilateral. Therefore, *ABCD* is a rhombus by Theorem 6b.

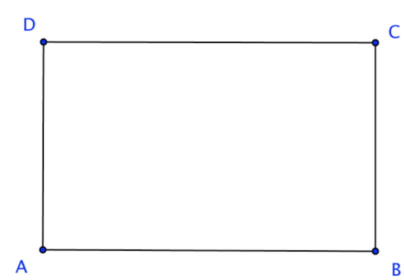
Now we are ready to prove Theorem 1e: If an angle bisector of a triangle is also a median, the triangle is isosceles.

Proof: In Δ*ABC*, ray *AD* bisects ∠*BAC* and *BD* = *CD*. Rotate Δ*ABC* 180˚ around *D*. Because *D* is the midpoint of *BC*, *B'* = *D* and *D'* = *B*. Because rotation preserves distance, *AD* = *A'D*. Now the diagonals of quadrilateral *ABA'C* bisect each other, so *ABA'C* is a parallelogram. But diagonal *AA'* bisects ∠*BAC*, so by Theorem 6e, *ABA'C* is a rhombus. A rhombus is equilateral, so *AB* = *AC*.



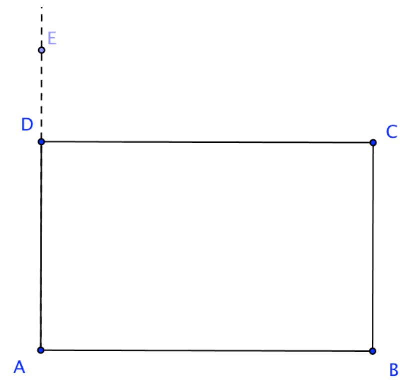
7. Definition of Rectangle: A quadrilateral with two lines of symmetry passing through midpoints of the opposite sides. (So, a rectangle is an isosceles trapezoid in two different ways.)

   a. If a quadrilateral is equiangular, it is a rectangle.
   Proof: Because ∠*A* = ∠*B* and ∠*C* = ∠*D*, ABCD is an isosceles trapezoid with line of symmetry through midpoints of *AB* and *DC* by Theorem 5b. Similarly, ∠*A* = ∠*D* and ∠*B* = ∠*C*, so *ABCD* is an isosceles trapezoid with line of symmetry through midpoints of *AD* and *BC*. By definition, *ABCD* is a rectangle.

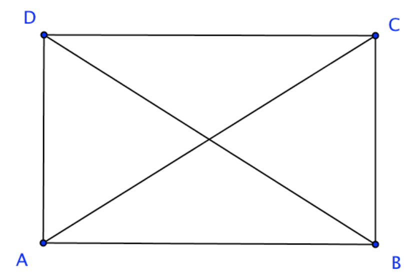


   b. If a parallelogram has a right angle, then the parallelogram is a rectangle.
   Proof: Suppose ∠*A* = 90˚. Then ∠*C* = 90˚ because opposite angles of a parallelogram are equal. The sum of the interior angles of a quadrilateral is 360˚, which leaves a total of 180˚ for ∠*B* and ∠*D*. Since they are also equal, they must be right angles as well. Hence all angles are equal right angles and the quadrilateral is a rectangle by Theorem 7a.

c. An isosceles trapezoid with a right angle is a rectangle.
Proof: Suppose $ABCD$ is an isosceles trapezoid with line of symmetry passing through bases AB and DC. Without loss of generality, we can suppose that $\angle A = 90^\circ$. Because the bases of an isosceles trapezoid are parallel $\angle EDC = \angle A = 90^\circ$. $\angle ADC$ and $\angle EDC$ are supplementary, so $\angle ADC = 90^\circ$ also. We also know that two consecutive angles of an isosceles trapezoid on the same base are equal, so $\angle B = \angle A = 90^\circ$and $\angle C = \angle ADC = 90^\circ$. Now $ABCD$ is equiangular, so by Theorem 7a it is a rectangle.



d. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.
Proof: Because $ABCD$ is a parallelogram, $AB \parallel DC$. Since the diagonals are equal as well, $ABCD$ is an isosceles trapezoid whose line of symmetry passes through midpoints of AB and DC by Theorem 5e. By the same argument with parallel sides AD and BC, $ABCD$ is an isosceles trapezoid whose line of symmetry passes through midpoints of AD and BC. It follows that $ABCD$ satisfies the symmetry definition of a rectangle.



8. Definition of Square: A quadrilateral with four lines of symmetry: two passing through opposite vertices and two passing through midpoints of opposite sides.

a. A rectangle with consecutive equal sides is a square.
Proof: A rectangle has two lines of symmetry passing through the opposite sides. The opposite sides of a rectangle are equal, so if two consecutive sides are also equal, it is equilateral. An equilateral quadrilateral is a rhombus, so its diagonals are additional lines of symmetry. Therefore, the rectangle is a square.

b. A rhombus with consecutive equal angles is a square.
Proof: The diagonals of a rhombus are lines of symmetry. The opposite angles of a rhombus are equal, so if two consecutive angles are also equal, it is equiangular. An equiangular quadrilateral is a rectangle, so it has two additional lines of symmetry passing through opposite sides. Therefore, the rhombus is a square

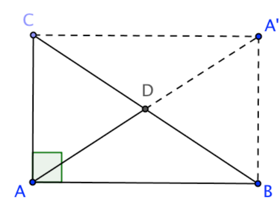
c. An equilateral quadrilateral with a right angle is a square.
Proof: An equilateral quadrilateral is a rhombus. Opposite angles of a quadrilateral are equal and the sum of the angles is 360˚, so all angles are right angles and the quadrilateral is also equiangular. An equiangular quadrilateral is a rectangle. If a quadrilateral is both a rhombus and a rectangle, it has four lines of symmetry and is therefore a square.

d. An equiangular quadrilateral with consecutive equal sides is a square.
Proof: An equiangular quadrilateral is a rectangle. The opposite sides of a rectangle are equal, and if consecutive sides are also equal, it must be equilateral. An equilateral quadrilateral is a rhombus. If a quadrilateral is both a rhombus and a rectangle, it has four lines of symmetry and is therefore a square.

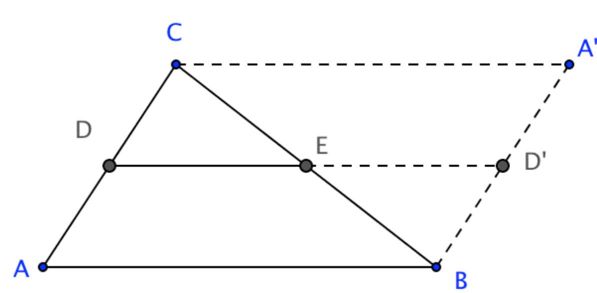
e. A quadrilateral with 4-fold rotational symmetry is a square.
Proof: Since a quadrilateral has four sides, consecutive sides and angles must map to each other under a 90˚ rotation. Because rotations preserve sides and angles, the quadrilateral must be both equilateral and equiangular, which implies that it is both a rectangle and a rhombus. If a quadrilateral is both a rhombus and a rectangle, it has four lines of symmetry and is therefore a square.

9. Additional Triangle Theorems

a. The median to the hypotenuse of a right triangle has half the length of the hypotenuse.
Proof: In right triangle *ABC*, *BD* = *CD*. Rotate Δ*ABC* and median *AD* 180˚ around *D*. Because rotations preserve segment length and the rotation is 180˚, *D* is the midpoint of *AA′* as well. Because the diagonals of quadrilateral *ABA′C* bisect each other, it is a parallelogram by Theorem 3a. But a parallelogram with a right angle is a rectangle, and the diagonals of a rectangle are equal. Thus *AD* = ½ *AA′* = ½ *BC*.



b. A segment joining the midpoints of two sides of a triangle (called a midsegment) is parallel to the third side and half as long.
Proof: In triangle *ABC*, *D* and *E* are midpoints of *AC* and *BC* respectively. Rotate Δ*ABC* and segment *DE* 180˚ around point *E*. Since *E* is a midpoint, *B′* = *C* and *C′* = *B*. Therefore, quadrilateral *ABA′C* has 2-fold rotational symmetry, so it is a parallelogram.



Because rotation preserves segment length and D is a midpoint, $AD = DC = D'B$. But AD is parallel to BD' as well, so $ABD'D$ is also a parallelogram by Theorem 3d. The opposite sides of a parallelogram are parallel, so $DE \parallel AB$. Because rotation preserves length, $DE = ED' = \frac{1}{2} DD'$, and because the opposite sides of a parallelogram are equal, $DD' = AB$. Hence $DE = \frac{1}{2} AB$.
Note: The proof is shorter and more elegant using dilation.